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On the Ising Model with Random Boundary Condition

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The infinite-volume limit behavior of the 2d Ising model under possibly strong random boundary conditions is studied. The model exhibits chaotic size-dependence at low temperatures and we prove that the '+' and '−' phases are the only almost sure limit Gibbs measures, assuming that the limit is taken along a sparse enough sequence of squares. In particular, we provide an argument to show that in a sufficiently large volume a typical spin configuration under a typical boundary condition contains no interfaces. In order to exclude mixtures as possible limit points, a detailed multi-scale contour analysis is performed.

KEY WORDS: Random boundary conditions; metastates; contour models; multi-scale analysis; local limit theorems.

1. INTRODUCTION

A fundamental problem in equilibrium statistical mechanics is to determine the set of physically accessible thermodynamic states for models defined via a family of local interactions. Usually[15,23] one interprets the extremal elements of the set of translationally invariant Gibbs measures as the pure thermodynamic phases of the model. In particular this means that one gathers all periodic or quasiperiodic extremal Gibbs measures into symmetry-equivalent classes and identifies the latter with the pure phases. Examples are the ferromagnetic, the antiferromagnetic, crystalline or quasicrystalline phases exhibited by various models. In this approach one does not consider either interface states or mixtures as pure phases. The mixtures allow for a unique decomposition into the extremal measures

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and are traditionally interpreted in terms of a lack of the knowledge about the thermodynamic state of the system. They can also be classified as less stable than the extremal measures.[25,36] It is thought that interface states which are extremal Gibbs measures are more stable than mixed states, but less so than pure phases. However, such an "intrinsic" characterization has not been developed. Note, moreover, that in disordered systems such as spin glasses, the stability of pure phases is *a priori* not clear and characterizing them remains an open question.

An efficient strategy for models with a simple enough structure of low-temperature phases is to associate these with suitable *coherent boundary conditions*. The latter are usually chosen as *ground states* of the model. As an example, the '+' and '−' Ising phases can be obtained by fixing the constant '+', respectively the constant '−' configurations at the boundaries and by letting the volume tend to infinity. This idea has been generalized to a wide class of models with both a finite and a 'mildly' infinite number of ground states, and is usually referred to as the *Pirogov–Sinai theory*.[4,11,45,46,48] The main assumption is that the different ground states are separated by high enough energy barriers, which can be described in terms of domain walls, referred to as contours. A useful criterion to check this so-called Peierls condition is within the formalism of m-potentials due to Holzstynski and Slawny[27].

An alternative strategy is to employ a boundary condition that does not favor any of the phases. Examples are the free and periodic boundary conditions for the zero-field Ising model, or the periodic boundary conditions for the Potts model at the critical temperature. In all these cases, an infinite-volume Gibbs measure is obtained that is a homogenous mixture of all phases.

Another scenario has been expected to occur for spin glasses. Namely, Newman and Stein have conjectured[37–39,41,42] that some spin glass models under symmetric boundary conditions exhibit non-convergence to a single thermodynamic limit measure, a phenomenon called *chaotic size-dependence* (see also refs. 14, 19, and 34). In this case, both the set of limit points of the sequence of the finite-volume Gibbs measures and their empirical frequency along the sequence of increasing volumes are of interest, and the formalism of *metastates* has been developed[39–41] to deal with these phenomena. These arguments have been made rigorous for a class of mean-field models,[7,8,17,30–32,43] whereas no such results are available for short-range spin glasses. For some general background on spin glasses and disordered models we refer to refs. 6, 20, 33, and 47.

A natural toy-problem where the usual contour methods can be used in the regime of chaotic size-dependence is the zero-field Ising model with the boundary condition sampled from a random distribution which is symmetric under the spin flip. In dimension 2 or more and at any subcritical temperature (including $T=0$) the finite-volume Gibbs measures are expected to oscillate randomly between the '+' and the '$-$' phases, demonstrating the chaotic size-dependence with exactly two limit points coinciding with the thermodynamic phases of the model.[38] In particular, one does not expect either any interface (e.g. Dobrushin) Gibbs states or any non-trivial statistical mixtures to occur as the limit points. This problem was addressed in ref. 16 where the conjecture was rigorously proven as the almost sure picture in the regime of the weak boundary coupling. In this regime, the boundary bonds are made sufficiently weaker w.r.t. the bulk bonds so that the interface configurations become damped exponentially with the size of the system, uniformly for *all* boundary conditions. Hence, all translationally non-invariant Gibbs measures are forbidden as possible limit points and one only needs to prove that the mixtures do not appear with probability 1.

In this paper we continue this study by removing the weakness assumption on the boundary bonds. To be specific, we consider the 2d Ising model with the random boundary condition sampled from the symmetric i.i.d. field $\{-1,1\}^{\mathbb{Z}^2}$ and coupled to the system via the *bulk* coupling constant. The conjecture remains true in this case and the crucial novelty of our approach is a detailed multi-scale analysis of contour models in the regime where realizations of the boundary condition are allowed that violate the "diluteness" (Peierls) condition, possibly making interfaces likely configurations. To be precise, these interfaces can have large Gibbs probabilities for certain boundary conditions, but we will show that such boundary conditions are sufficiently unlikely to occur for large volumes. An important side-result is the almost sure absence of interface configurations. This means that for a typical boundary condition, the probability of the set of configurations containing an interface tends to zero in the infinite-volume limit. Note that this excludes interfaces in a stronger way than the familiar result about the absence of translationally non-invariant Gibbs measures in the 2d Ising model.[1,22,26] Indeed, the absence of fluctuating interfaces basically means that not only the expectations of local functions but also their space averages (e.g. the volume-averaged magnetization) have only two limit points, corresponding to the two Ising phases. Hence, we believe that our techniques allow for a natural generalization to any dimension $d\geqslant 2$. However, as already argued in ref. 16, in dimensions $d\geqslant 4$, the set $\{\mu^+,\mu^-\}$ is expected (and partially proven) to be the *almost sure* set of limit measures, the limit being taken along the regular sequence of cubes. On the other hand, for $d=2,3$ the same result can only be obtained if the limit is taken along a sparse enough sequence of cubes. In the latter case it remains an open problem to analyze the set of limit points along the regular sequence of cubes. Our conjecture is that the almost sure set of limit points coincides then with the set of all translationally invariant Gibbs measures, i.e. including the mixtures.

The structure of the paper is as follows. We will first introduce our notation in Section 2, and describe our results in Section 3. Then in Sections 4 and 5 we will introduce a contour representation of the model and set up our cluster expansion formalism. In Section 6 we first exclude the occurrence of interfaces. In the rest of the paper we develop a multi-scale argument, providing a weak version of the local limit theorem to show that no mixed states can occur as limit points in the infinite-volume limit. Two general results, the first one on a variant of the cluster expansion convergence criteria for polymer models and the second one on local limit upper bounds, are collected in Appendices A and B.

2. SET-UP

We consider the two-dimensional square lattice $\mathbb{Z}^2$ and use the symbols $\sigma, \eta, \ldots$ for the maps $\mathbb{Z}^2 \mapsto \{-1, 1\}$. They are called *spin configurations* and the set of all spin configurations is $\Omega = \{-1, 1\}^{\mathbb{Z}^2}$. Furthermore, the symbol σ_A is used for the restriction of a spin configuration $\sigma \in \Omega$ to the set $A \subset \mathbb{Z}^2$. If $A = \{x\}$, we write σ_x instead. The set of all restrictions of Ω to the set A is Ω_A.

A function $f : \Omega \mapsto \mathbb{R}$ is called *local* whenever there is a finite set $D \subset \mathbb{Z}^2$ such that $\sigma_D = \sigma'_D$ implies $f(\sigma) = f(\sigma')$. The smallest set with this property is called the *dependence set* of the function f and we use the symbol $\mathcal{D}_f$ for it. To every local function f we assign the supremum norm $\|f\| = \sup_{\sigma \in \Omega} |f(\sigma)|$.

The spin configuration space Ω comes equipped with the product topology, which is followed by the weak topology on the space $M(\Omega)$ of all probability measures on Ω. The latter is introduced via the collection of seminorms

$$\|\mu\|_X = \sup_{\substack{\|f\|=1 \\ \mathcal{D}_f \subset X}} |\mu(f)| \tag{1}$$

upon all finite $X \subset \mathbb{Z}^2$. Then, the weak topology is generated by the collection of open balls $B_X^{\epsilon}(\mu) = \{\nu;\ \|\nu - \mu\|_X < \epsilon\}$, $\epsilon > 0$, X finite, and a sequence $\mu_n \in M(\Omega)$ weakly converges to μ if and only if $\|\mu_n - \mu\|_X \to 0$ for all finite $X \subset \mathbb{Z}^2$. Under the weak topology, $M(\Omega)$ is compact.

We consider a collection of the Hamiltonians $H_{\Lambda}^{\eta} : \Omega_{\Lambda} \mapsto \mathbb{R}$ for all square volumes $\Lambda = \Lambda(N)$, $N = 1, 2, \ldots$,

$$\Lambda(N) = \{x \in \mathbb{Z}^2;\ \|x\| \leqslant N\}, \quad \|x\| = \max\{|x_1|, |x_2|\} \tag{2}$$

and *boundary conditions* $\eta \in \Omega$. The Hamiltonians are given by

$$H^{\eta}_{\Lambda}(\sigma_{\Lambda}) = -\beta \sum_{\langle x,y\rangle \subset \Lambda} (\sigma_x \sigma_y - 1) - \beta \sum_{\substack{\langle x,y\rangle \\ x\in\Lambda,\, y\in\Lambda^c}} \sigma_x \eta_y \tag{3}$$

where $\langle x, y\rangle$ stands for pairs of nearest neighboring sites, i.e. such that $\|x-y\|_1 := |x_1 - y_1| + |x_2 - y_2| = 1$, and $\Lambda^c = \mathbb{Z}^2 \setminus \Lambda$. We consider the ferromagnetic case, $\beta > 0$. Following a familiar framework, we introduce the *finite-volume Gibbs measure* $\mu^{\eta}_{\Lambda} \in M(\Omega)$ by

$$\mu^{\eta}_{\Lambda}(\sigma) = \frac{1}{Z^{\eta}_{\Lambda}} \exp[-H^{\eta}_{\Lambda}(\sigma_{\Lambda})]\, 1\!\!1_{\{\sigma_{\Lambda^c} = \eta_{\Lambda^c}\}} \tag{4}$$

and define the set $\mathcal{G}_{\beta}$ of (infinite-volume) *Gibbs measures*, $\mathcal{G}_{\beta}$, as the weak closure of the convex hull over the set of all weak limit points of the sequences $(\mu^{\eta}_{\Lambda(N)})_{N\to\infty}$, $\eta \in \Omega$. A standard result reads that there exists β_c such that for any $\beta > \beta_c$ the set of Gibbs measures $\mathcal{G}_{\beta} = \{\alpha\mu^+ + (1-\alpha)\mu^-;\ 0 \leqslant \alpha \leqslant 1\}$. Here, the extremal measures $\mu^{\pm}$ are translation-invariant, they satisfy the symmetry relation $\int d\mu^+(\sigma)\, f(\sigma) = \int d\mu^-(\sigma)\, f(-\sigma)$, and can be obtained as the weak limits $\lim_{N\to\infty} \mu^{\eta}_{\Lambda(N)}$ for $\eta \equiv \pm 1$.

3. RESULTS

We consider the limit behavior of the sequence of finite-volume Gibbs measures $(\mu^{\eta}_{\Lambda(N)})_{N\in\mathbb{N}}$ under boundary conditions η sampled from the i.i.d. symmetric random field

$$\boldsymbol{P}\{\eta_x = 1\} = \boldsymbol{P}\{\eta_x = -1\} = \frac{1}{2} \tag{5}$$

Our first result concerns the almost sure structure of the set of all limit points of the sequence of the finite-volume Gibbs measures, the limit being taken along a sparse enough sequence of squares.

Theorem 3.1. For arbitrary $\omega > 0$ there is a $\beta_1 = \beta_1(\omega)$ such that for any $\beta \geqslant \beta_1$ the set of all weak limit points of any sequence $(\mu_{\Lambda(k_N)})_{N=1,2,\ldots}$, $k_N \geqslant N^{2+\omega}$, is $\{\mu^+, \mu^-\}$, $\boldsymbol{P}$-a.s.

Remark 3.2. The above theorem does not exclude other measures as the almost sure limit points, provided that other (non-sparse) sequences of squares are taken instead. Actually, our conjecture is that, for β large enough, the set of all weak limit points of $(\mu_{\Lambda(N)})_{N=1,2,\ldots}$ coincides $\boldsymbol{P}$-a.s.

with $\mathcal{G}_\beta$. On the other hand, in dimension 3, it is rather expected to coincide with the set of all translation-invariant Gibbs measures, and, in any dimension higher than 3, with the set $\{\mu^+, \mu^-\}$.

Remark 3.3. A modification of the Hamiltonian (3) is obtained by re-scaling the boundary coupling by a factor λ to get

$$H_\Lambda^{\lambda,\eta}(\sigma_\Lambda) = -\beta \sum_{\langle x,y\rangle\subset\Lambda} (\sigma_x\sigma_y - 1) - \lambda\beta \sum_{\substack{\langle x,y\rangle \\ x\in\Lambda,\, y\in\Lambda^c}} \sigma_x \eta_y \tag{6}$$

In this case, the claim of Theorem 3.1 for the sequence of the finite-volume Gibbs measures

$$\mu_\Lambda^{\lambda,\eta}(\sigma) = \frac{1}{\mathcal{Z}_\Lambda^{\lambda,\eta}} \exp[-H_\Lambda^{\lambda,\eta}(\sigma_\Lambda)]\, 1\!\!1_{\{\sigma_{\Lambda^c}=\eta_{\Lambda^c}\}} \tag{7}$$

was proven in ref. 16 under the condition that $|\lambda|$ is small enough (= the boundary coupling is sufficiently weak w.r.t. the bulk one). It was also shown that $\{\mu^+, \mu^-\}$ is the almost sure set of limit points of the sequence $(\mu^\eta_{\Lambda(N)})_{N\in\mathbb{N}}$, provided that the space dimension is at least 4.

To reveal the nature of all possible limit points that can appear along the sequence of squares $\Lambda(N)$, $N = 1, 2, \ldots$, we study the empirical frequency for the finite-volume Gibbs states from the sequence $(\mu^\eta_{\Lambda(N)})_{N\in\mathbb{N}}$ to occur in a fixed set of measures. More precisely, for any set $B \subset M(\Omega)$, boundary condition $\eta\in\Omega$, and $N = 1, 2, \ldots$, we define

$$Q_N^{B,\eta} = \frac{1}{N} \sum_{k=1}^{N} 1\!\!1_{\{\mu^\eta_{\Lambda(k)}\in B\}} \tag{8}$$

The next theorem shows the *null-recurrent* character of all measures different from both μ^+ and μ^-. We use the notation $\bar{B}$ and B^0 for the weak closure and the weak interior of B, respectively.

Theorem 3.4. There is β_2 such that for any $\beta \geqslant \beta_2$ and any set $B \subset M(\Omega)$, one has

$$\lim_{N\uparrow\infty} Q_N^{B,\eta} = \begin{cases} 0 & \text{if } \mu^+, \mu^- \notin \bar{B} \\ \frac{1}{2} & \text{if } \mu^\pm \in B^0 \text{ and } \mu^\mp \notin \bar{B} \\ 1 & \text{if } \mu^+, \mu^- \in B^0 \end{cases} \tag{9}$$

with $\boldsymbol{P}$-probability 1.

Both theorems follow in a straightforward way from the following key estimate that will be proven in the sequel of the paper.

Proposition 3.5. Given $\alpha > 0$, there is a $\beta_0 = \beta_0(\alpha)$ such that for any $\beta \geqslant \beta_0$, $\epsilon > 0$ and $X \subset \mathbb{Z}^d$ finite,

$$\overline{\lim_{N\to\infty}}\, N^{\frac{1}{2}-\alpha}\, \boldsymbol{P}\{(\|\mu^{\eta}_{\Lambda(N)} - \mu^{+}\|_X \wedge \|\mu^{\eta}_{\Lambda(N)} - \mu^{-}\|_X) \geqslant \epsilon\} < \infty \tag{10}$$

Remark 3.6. The proposition claims that, for a *typical* $\eta \in \Omega$, the finite-volume Gibbs measures are expected to be near the extremal Gibbs measures $\mu^{\pm}$. The above probability upper-bound of the form $\mathcal{O}\left(N^{-\frac{1}{2}+\alpha}\right)$ will be proven by means of a variant of the local limit theorem for the sum of weakly dependent random variables. Although we conjecture the correct asymptotics to be of order $N^{-\frac{1}{2}}$, the proof of *any* lower bound goes beyond the presented technique. This is why the detailed structure of the almost sure set of the limit Gibbs measures is not available, except for the limits taken along sparse enough sequences of squares.

Proof of Theorem 3.1. Given $\omega > 0$, we choose an $\alpha < \omega/(2(2+\omega))$ and define $\beta_1(\omega) = \beta_0(\alpha)$. Let $\beta \geqslant \beta_1(\omega)$ and $k_N \geqslant N^{2+\omega}$.

First let $\mu \notin \{\mu^{+}, \mu^{-}\}$. There exists a weakly open set $B \subset M(\Omega)$ such that $\mu \in B$ and $\mu^{+}, \mu^{-} \notin \bar{B}$. Choosing a finite set $X \subset \mathbb{Z}^2$ and $\epsilon > 0$ such that $B^{\epsilon}_X(\mu^{\pm}) \cap B = \emptyset$, Proposition 3.5 gives the bound

$$\begin{aligned}\boldsymbol{P}\{\mu^{\eta}_{\Lambda(k_N)} \in B\} &\leqslant \boldsymbol{P}\{\mu^{\eta}_{\Lambda(k_N)} \notin B^{\epsilon}_X(\mu^{+}) \cup B^{\epsilon}_X(\mu^{-})\}\\ &= \mathcal{O}(k(N)^{-\frac{1}{2}+\alpha}) = \mathcal{O}(N^{-1+\alpha(2+\omega)-\frac{\omega}{2}})\end{aligned} \tag{11}$$

Since $\sum_N \boldsymbol{P}\{\mu^{\eta}_{\Lambda(k_N)} \in B\} < \infty$, the set B contains $\boldsymbol{P}$-a.s. no limit points of the sequence $\mu^{\eta}_{\Lambda(k_N)}$ due to the Borel–Cantelli argument. Hence, with $\boldsymbol{P}$-probability 1, μ is not a limit point.

To prove that both μ^{+} and μ^{-} are $\boldsymbol{P}$-a.s. limit points, take any finite set of sites X and $\epsilon > 0$ such that $B^{\epsilon}_X(\mu^{+}) \cap B^{\epsilon}_X(\mu^{-}) = \emptyset$. By the symmetry of the distribution, $\boldsymbol{P}\{\mu_{\Lambda(k_N)} \in B^{\epsilon}_X(\mu^{+})\} = \boldsymbol{P}\{\mu_{\Lambda(k_N)} \in B^{\epsilon}_X(\mu^{-})\}$ and, employing Proposition 3.5 again, $\lim_N \boldsymbol{P}\{\mu_{\Lambda(k_N)} \in B^{\epsilon}_X(\mu^{\pm})\} = \frac{1}{2}$. By the Borel–Cantelli and the compactness arguments, the weak closure $\bar{B}^{\epsilon}_X(\mu^{\pm})$ contains a limit point, $\boldsymbol{P}$-a.s. As $\mu^{\pm} = \cap_{X,\epsilon} \bar{B}^{\epsilon}_X(\mu^{\pm})$, the statement is proven. ∎

Proof of Theorem 3.4. Choose $\beta_2 = \beta_0(\alpha)$ for an arbitrary $\alpha \in (0, \frac{1}{2})$ and assume $\beta \geqslant \beta_2$, $B \in M(\Omega)$. Using the notation $q^{B,\eta}_N = \boldsymbol{P}\{\mu^{\eta}_{\Lambda(N)} \in B\}$ and repeating the reasoning in the proof of Theorem 3.1, one gets

$$\begin{aligned} \boldsymbol{E}\, 1\!\!1_{\{\mu^{\eta}_{\Lambda(N)} \in B\}} &= q_N^{B,\eta} \\ &= \begin{cases} \mathcal{O}(N^{-\frac{1}{2}+\alpha}) \to 0 & \text{if } \mu^+, \mu^- \notin \bar{B} \\ \frac{1}{2} - \mathcal{O}(N^{-\frac{1}{2}+\alpha}) \to \frac{1}{2} & \text{if } \mu^\pm \in B^0 \text{ and } \mu^\mp \notin \bar{B} \\ 1 - \mathcal{O}(N^{-\frac{1}{2}+\alpha}) \to 1 & \text{if } \mu^\pm \in B^0 \end{cases} \end{aligned} \tag{12}$$

and

$$\mathrm{Var}\, 1\!\!1_{\{\mu^{\eta}_{\Lambda(N)} \in B\}} = q_N^{B,\eta}(1 - q_N^{B,\eta}) \leqslant \frac{1}{4} \tag{13}$$

Hence, $\sum_N \frac{1}{N^2} \mathrm{Var}\, 1\!\!1_{\{\mu^{\eta}_{\Lambda(N)} \in B\}} < \infty$ and since the functions $1\!\!1_{\{\mu^{\eta}_{\Lambda(N)} \in B\}}$, $N = 1, 2, \ldots$ are independent, the result immediately follows from the strong law of large numbers.[13] ▮

4. GEOMETRICAL REPRESENTATION OF THE MODEL

We define the *dual lattice* $(\mathbb{Z}^2)^* = \mathbb{Z}^2 + (1/2, 1/2)$. The (unordered) pairs of nearest neighboring sites $\langle x, y \rangle \subset \mathbb{Z}^2$ are called *bonds* and to every bond we assign a unique *dual bond* $\langle x^*, y^* \rangle \equiv \langle x, y \rangle^* \subset (\mathbb{Z}^2)^*$. Given a set of dual bonds A^*, we use the symbol $|A^*|$ to denote the number of all dual bonds in A^*. Further, with a slight abuse of notation, we also write $x^* \in A^*$ whenever there exists a dual bond $\langle x^*, y^* \rangle \in A^*$, i.e. A^* also stands for the corresponding set of dual sites.

Any set A^* of dual bonds is called *connected* whenever for any dual sites $x^*, y^* \in A^*$ there exists a sequence of dual bonds $\langle x^*, x_1^* \rangle, \langle x_1^*, x_2^* \rangle, \ldots, \langle x_{k-1}^*, y^* \rangle \in A^*$. The distance $d[A^*, B^*]$ of the sets of dual bonds A^*, B^* is defined as the smallest integer k such that there exist $x^* \in A^*$, $y^* \in B^*$, and a sequence of dual bonds $\langle x^*, x_1^* \rangle, \langle x_1^*, x_2^* \rangle, \ldots, \langle x_{k-1}^*, y^* \rangle \subset (\mathbb{Z}^2)^*$. Similarly, a set of sites $A \subset \mathbb{Z}^2$ is called *connected* whenever for all $x, y \in A$ there exists a sequence of bonds $\langle x, x_1 \rangle, \langle x_1, x_2 \rangle, \ldots, \langle x_{k-1}, y \rangle \subset A$. Correspondingly, the distance $d[A, B]$ of the sets $A, B \subset \mathbb{Z}^2$ is understood in the sense of the $\|.\|_1$-norm.

In the sequel we assume that a volume $\Lambda = \Lambda(N)$ is fixed and we define the *boundary* $\partial\Lambda$ as the set of all dual bonds $\langle x, y \rangle^*$ such that $x \in \Lambda$ and $y \in \Lambda^c$. In general, ∂A, $A \subset \Lambda$ is the set of all dual bonds $\langle x, y \rangle^*$, $x \in A$, $y \in \Lambda^c$. For any subset $P \subset \partial\Lambda$ we use the symbol $\underline{P}$ to denote the set of all sites $y \in \Lambda^c$ such that there is a (unique) bond $\langle x, y \rangle^* \in P$, $x \in \Lambda$. If $\underline{P}$ is a connected set of sites, then P is called a *boundary interval*. Obviously, any boundary interval is a connected set of dual bonds, however, the opposite is not true. However, any set $P \subset \partial\Lambda$ has a unique decomposition into a family of (maximal) boundary intervals. Furthermore, consider all connected sets P_i of dual bonds satisfying $P \subset P_i \subset \partial\Lambda$ which are minimal in the sense of inclusion. The smallest of these sets is called $\mathrm{Con}(P)$ (in the case of an equal size take the first one in the lexicographic order) and we use the shorthand $|P| = |\mathrm{Con}(P)|$. Finally, we define the *corners* of $\Lambda(N)$ as the dual sites $x^*_{C,1} = (-N-1/2, -N-1/2)$, $x^*_{C,2} = (N+1/2, -N-1/2)$, $x^*_{C,3} = (N+1/2, N+1/2)$, and $x^*_{C,4} = (-N-1/2, N+1/2)$.

Pre-contours. Given a configuration $\sigma \in \Omega_\Lambda = \{-1, +1\}^\Lambda$, the dual bond $\langle x, y\rangle^*$ to a bond $\langle x, y\rangle \subset \Lambda$ is called *broken* whenever $\sigma_x \neq \sigma_y$, and the set of all the broken dual bonds is denoted by $\Delta_\Lambda(\sigma)$. In order to define a suitable decomposition of the set $\Delta_\Lambda(\sigma)$ into components, we take advantage of a certain freedom in such a construction to obtain the components with suitable geometrical properties. In this first step, we define the *pre-contours* as follows. Consider all maximal connected components of the set of dual bonds $\Delta_\Lambda(\sigma)$. By the standard "rounding-corner" procedure, see Fig. 1, we further split them into connected (not necessarily disjoint) subsets, γ, which can be identified with (open or closed) simple curves. Namely,

$$\gamma = \{\langle x^*_0, x^*_1\rangle, \langle x^*_1, x^*_2\rangle, \dots, \langle x^*_{k-1}, x^*_k\rangle\}, \quad k \in \mathbb{N}$$

such that if $x^*_i = x^*_j$, $i \neq j$, then $\{i, j\} = \{0, k\}$ and γ is closed. Otherwise, $x^*_i \neq x^*_j$ for all $i \neq j$ and γ is open with $x^*_0, x^*_k \in \partial\Lambda$.

These γ are called *pre-contours* and we use the symbol $\tilde{\mathcal{D}}_\Lambda(\sigma)$ for the set of all pre-contours corresponding to σ; write also $\tilde{\mathcal{D}}_\Lambda = \{\tilde{\mathcal{D}}_\Lambda(\sigma), \sigma \in \Omega_\Lambda\}$ and use the symbol $\tilde{\mathcal{K}}_\Lambda$ for the set of all pre-contours in Λ. Any pair of pre-contours $\gamma_1, \gamma_2 \in \tilde{\mathcal{K}}_\Lambda$ is called *compatible* whenever there is a configuration $\sigma \in \Omega_\Lambda$ such that $\gamma_1, \gamma_2 \in \tilde{\mathcal{D}}_\Lambda(\sigma)$. A set of pairwise compatible pre-contours is called a *compatible set*. Obviously, $\tilde{\mathcal{D}}_\Lambda$ is simply the collection of all compatible sets of pre-contours from $\tilde{\mathcal{K}}_\Lambda$. Intuitively, the pre-contours that are

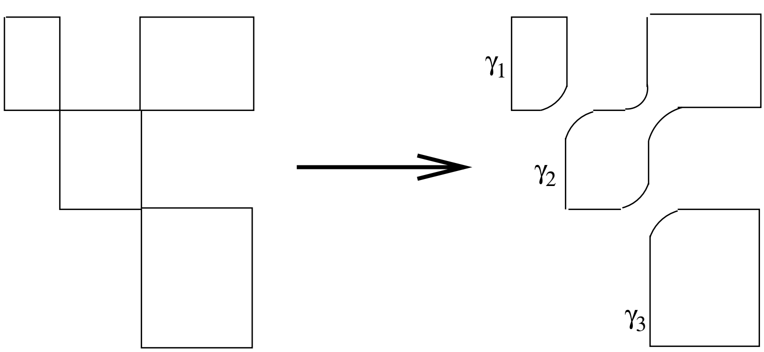

Fig. 1. Pre-contours constructed via the rounding-corner procedure.

closed curves coincide with the familiar Ising contours, whereas the pre-contours touching the boundary become open curves.

Obviously, $\Omega_\Lambda \mapsto \tilde{\mathcal{D}}_\Lambda$ is a two-to-one map with the images of the configurations σ and $-\sigma$ being identical. In order to further analyze this map, we introduce the concept of interior and exterior of the pre-contours briefly as follows (the details can be found in refs. 5 and 6). If $\sigma \in \Omega_\Lambda$ is a configuration such that $\tilde{\mathcal{D}}_\Lambda(\sigma) = \{\gamma\}$, then there is a unique decomposition of the set Λ into a pair of disjoint connected subsets, $\Lambda = \Lambda_1 \cup \Lambda_2$, such that for any bond $\langle x, y\rangle$, $x \in \Lambda_1$, $y \in \Lambda_2$, one has $\langle x, y\rangle^* \in \gamma$. These are called the *exterior*, $\mathrm{Ext}(\gamma)$, and the *interior*, $\mathrm{Int}(\gamma)$, where the assignment is given by the following procedure. We distinguish three mutually exclusive classes of pre-contours:

(i) *Bulk pre-contours*
$\partial\Lambda = \partial\Lambda_1$. Then, $\mathrm{Ext}(\gamma) := \Lambda_1$ and $\mathrm{Int}(\gamma) := \Lambda_2$, see Fig. 1.

(ii) *Small boundary pre-contours*
Λ_1 contains at least three corners of Λ and $\partial\Lambda_2 \neq \emptyset$. Then, $\mathrm{Ext}(\gamma) := \Lambda_1$ and $\mathrm{Int}(\gamma) := \Lambda_2$, see Fig. 2.

(iii) *Interfaces*
Both Λ_1 and Λ_2 contain exactly two corners of Λ and (a) $|\Lambda_1| > |\Lambda_2|$, or (b) $|\Lambda_1| = |\Lambda_2|$ and $x_{C,1} \in \Lambda_1$. Then, $\mathrm{Ext}(\gamma) := \Lambda_1$ and $\mathrm{Int}(\gamma) := \Lambda_2$, see Fig. 3.

The set $\partial\gamma := \partial\mathrm{Int}(\gamma)$ is called the *boundary* of the pre-contour γ.

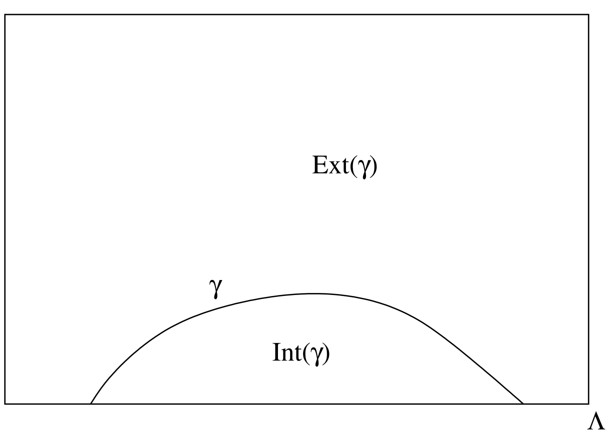

Fig. 2. Small boundary pre-contour.

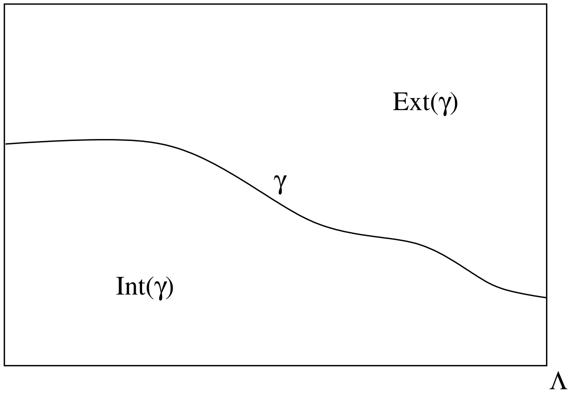

Fig. 3. Interface.

Contours. Next, we define contours by gluing some boundary pre-contours together via the following procedure. Any compatible pair of pre-contours $\gamma_1, \gamma_2 \in \tilde{\mathcal{K}}_\Lambda$ is called *boundary-matching* iff $\partial\gamma_1 \cap \partial\gamma_2 \neq \emptyset$. Any compatible set of pre-contours such that the graph on this set obtained by connecting the pairs of boundary-matching pre-contours becomes connected is called a *contour*. In particular, every bulk pre-contour is boundary-matching with no other compatible pre-contour. Therefore, every bulk pre-contour is trivially a contour. We use the symbol $\mathcal{D}_\Lambda(\sigma)$ for the set of all contours corresponding to $\sigma \in \Omega_\Lambda$ and $\mathcal{K}_\Lambda$ for the set of all contours in Λ. Any pair of contours Γ_1, Γ_2 is compatible, $\Gamma_1 \sim \Gamma_2$, whenever all pairs of pre-contours $\gamma_1 \in \Gamma_1$, $\gamma_2 \in \Gamma_2$ are compatible, and we write $\mathcal{D}_\Lambda$ for the set of all families of pairwise compatible contours in Λ. All the above geometrical notions naturally carry over to contours and we define the exterior, $\text{Ext}(\Gamma) := \cap_{\gamma\in\Gamma}\text{Ext}(\gamma)$, the interior, $\text{Int}(\Gamma) := \Lambda \setminus \text{Ext}(\Gamma)$ (in general, not a connected set anymore), the boundary $\partial\Gamma := \cup_{\gamma\in\Gamma}\partial\gamma$, and the length $|\Gamma| := \sum_{\gamma\in\Gamma}|\gamma|$. Similarly, if $\partial \in \mathcal{D}_\Lambda$ is a configuration of contours, let $\text{Ext}(\partial) := \cap_{\Gamma\in\partial}\text{Ext}(\Gamma)$, $\text{Int}(\partial) := \Lambda \setminus \text{Ext}(\partial)$, and $|\partial| := \sum_{\Gamma\in\partial}|\Gamma|$.

Eventually we arrive at the following picture. The set $\mathcal{K}_\Lambda$ of contours is a union of three disjoint sets of contours, namely of the sets of all

(i) *Bulk (pre-)contours.*

(ii) *Small boundary contours* Γ defined by 1) $\partial\Gamma \neq \emptyset$, and 2) no pre-contour $\gamma \in \Gamma$ is an interface.

 (a) *Simple* small boundary contours: the boundary $\partial\Gamma$ contains no corner, i.e. $\partial\Gamma$ is a boundary interval.

(b) *Corner* small boundary contours: there is exactly one corner $x^*_{C,i} \in \partial\Gamma$.

(iii) *Large boundary contours* Γ, i.e. containing at least one interface $\gamma \in \Gamma$.

Examples of the bulk, small boundary, and large boundary contours are given in Figs. 4–5.

Furthermore, $\mathcal{D}_\Lambda(\sigma)$ is a two-to-one map $\Omega_\Lambda \mapsto \mathcal{D}_\Lambda$ satisfying the spin-flip symmetry $\mathcal{D}_\Lambda(\sigma) = \mathcal{D}_\Lambda(-\sigma)$. Since σ takes a unique spin value in the set $\mathrm{Ext}(\mathcal{D}_\Lambda(\sigma))$, there is a natural decomposition $\Omega_\Lambda = \Omega^+_\Lambda \cup \Omega^-_\Lambda$ according to this value, i.e.

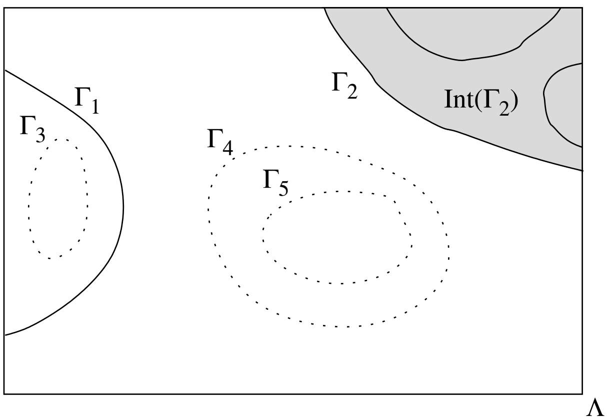

Fig. 4. Bulk and small boundary contours.

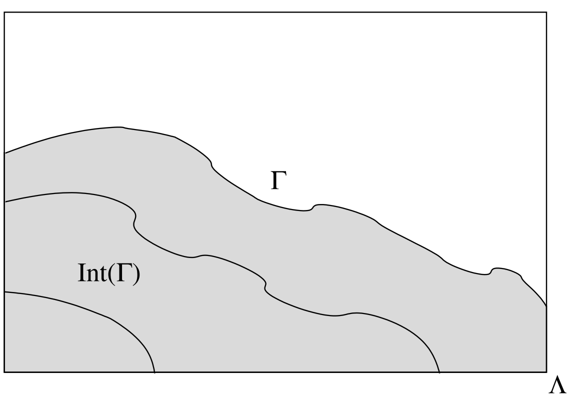

Fig. 5. Large boundary contour.

$$\Omega_\Lambda^\pm := \{\sigma \in \Omega_\Lambda;\ \sigma|_{\mathrm{Ext}(\mathcal{D}_\Lambda(\sigma))} = \pm 1\} = -\Omega_\Lambda^\mp \tag{14}$$

As a consequence, $\mathcal{D}_\Lambda$ splits into a conjugated (by spin-flip symmetry) pair of one-to-one maps $\Omega_\Lambda^\pm \mapsto \mathcal{D}_\Lambda$. This enables us to represent the finite-volume Gibbs measure (4) in the form of a convex combination of two conjugated constrained Gibbs measures as follows:

$$\mu_\Lambda^\eta(\sigma) = \left[1 + \frac{\mathcal{Z}_\Lambda^{-,\eta}}{\mathcal{Z}_\Lambda^{+,\eta}}\right]^{-1} \nu_\Lambda^{+,\eta}(\sigma) + \left[1 + \frac{\mathcal{Z}_\Lambda^{+,\eta}}{\mathcal{Z}_\Lambda^{-,\eta}}\right]^{-1} \nu_\Lambda^{-,\eta}(\sigma) \tag{15}$$

where we have introduced the Gibbs measure constrained to $\Omega_\Lambda^\pm$ by

$$\nu_\Lambda^{\pm,\eta}(\sigma) = \frac{1}{\mathcal{Z}_\Lambda^{\pm,\eta}} \exp[-H_\Lambda^\eta(\sigma)]\, \mathbf{1}_{\{\sigma \in \Omega_\Lambda^\pm\}} \tag{16}$$

Moreover, for any $\sigma \in \Omega_\Lambda^\pm$, the Hamiltonian can be written as

$$H_\Lambda^\eta(\sigma) = E_\Lambda^{\pm,\eta}(\partial) + 2\beta \sum_{\Gamma \in \partial} |\Gamma| \tag{17}$$

with $\partial = \mathcal{D}_\Lambda(\sigma)$, and we have introduced

$$E_\Lambda^{\pm,\eta}(\partial) = -\beta \sum_{\substack{\langle x,y \rangle \\ x \in \Lambda,\, y \in \Lambda^c}} \sigma_x \eta_y \tag{18}$$

Finally, $\mathcal{Z}_\Lambda^{\pm,\eta}$ is essentially the partition function of a polymer model,[29] see also Appendix A,

$$\mathcal{Z}_\Lambda^{\pm,\eta} = \exp\left(-E_\Lambda^{\pm,\eta}(\emptyset)\right) \textstyle\sum_{\partial \in \mathrm{D}_\Lambda} \prod_{\Gamma \in \partial} \rho^{\pm,\eta}(\Gamma) \tag{19}$$

where the polymers coincide with the contours and the polymer weights are defined by

$$\rho^{\pm,\eta}(\Gamma) = \exp\left(-2\beta|\Gamma|\right) \exp\left(-E^{\pm,\eta}(\Gamma) + E^{\pm,\eta}(\emptyset)\right) \tag{20}$$

By the spin-flip symmetry, we can confine ourselves to the '+' case and use the shorthand notations $\rho^\eta(\Gamma) := \rho^{+,\eta}(\Gamma) = \rho^{-,-\eta}(\Gamma)$, $\mathcal{Z}_\Lambda^\eta := \mathcal{Z}_\Lambda^{+,\eta} = \mathcal{Z}_\Lambda^{-,-\eta}$, $E_\Lambda^\eta := E_\Lambda^{+,\eta} = E_\Lambda^{-,-\eta}$, and $\nu_\Lambda^\eta(\sigma) := \nu_\Lambda^{+,\eta}(\sigma) = \nu_\Lambda^{-,-\eta}(-\sigma)$. Moreover, the boundary $\partial\Gamma$ of a contour Γ has a natural decomposition into components as follows. Let $\sigma \in \Omega_\Lambda^+$ be such that $\mathcal{D}_\Lambda(\sigma) = \{\Gamma\}$. Then the '$\pm$' boundary component $\partial\Gamma^\pm$ is defined as the set of all dual bonds $\langle x, y\rangle^*$ such that $x \in \Lambda$, $y \in \Lambda^c$, $\sigma_x = \pm 1$. With this definition, the contour weight (20) is

$$\rho^\eta(\Gamma) = \exp\left[-2\beta\left(|\Gamma| + \sum_{x \in \underline{\partial\Gamma^-}} \eta_x\right)\right] \tag{21}$$

Using the representation (15) of the finite-volume Gibbs measure μ_Λ^η, the strategy of our proof consists of two main parts:

(1) To prove that the constrained (random) Gibbs measure ν_Λ^η asymptotically coincides with the Ising '+' phase, for almost all η.

(2) To show that a sufficiently sparse subsequence of the sequence of random free energy differences $\log \mathcal{Z}_\Lambda^\eta - \log \mathcal{Z}_\Lambda^{-\eta}$ has $+\infty$ and $-\infty$ as the only limit points, for almost all η.

Then, Proposition 3.5 follows almost immediately. Moreover, we will show that for a $\boldsymbol{P}$-typical boundary condition η and a μ_Λ^η-typical configuration $\sigma \in \Omega_\Lambda$, the corresponding set of pre-contours $\tilde{\mathcal{D}}_\Lambda(\sigma)$ contains no interfaces.

Theorem 4.1. There is β_3 such that for any $\beta \geqslant \beta_3$ one has

$$\lim_{N\uparrow\infty} \mu_{\Lambda(N)}^\eta\{\tilde{\mathcal{D}}_{\Lambda(N)}(\sigma) \text{ contains an interface}\} = 0 \tag{22}$$

for $\boldsymbol{P}$-a.e. $\eta \in \Omega$.

Remark 4.2. Note that the low-temperature result by Gallavotti[22], extended to all subcritical temperatures in refs. 1 and 26, about the absence of translationally non-invariant Gibbs measures in the 2d Ising model does not exclude fluctuating interfaces under a suitably arranged ("Dobrushin-like") boundary condition. On the other hand, the above theorem claims that a typical boundary condition gives rise to a Gibbs measure in which interfaces anywhere are suppressed. We mention this side-result to demonstrate the robustness of the presented multi-scale approach and to argue that it is essentially dimension-independent, the $d = 2$ case being chosen only for simplicity.

It is easy to realize that, for a typical η, the polymer model (19) fails the "diluteness" condition on the sufficient exponential decay of the polymer weights, which means one cannot directly apply the familiar formalism of cluster expansions. These violations of the diluteness condition occur locally along the boundary with low probability, and hence have typically low densities. Nevertheless, their presence on all scales forces a sequential, multi-scale, treatment. Multi-scale methods have been employed at various occasions, such as for one-phase models in the presence of Griffiths singularities or for the random field Ising model.[9,10,12,18,21,28] In contrast to the usual case of cluster expansions one does not obtain analyticity (which may not even be valid). In our approach, we loosely follow the ideas of Fröhlich and Imbrie.[21] For other recent work developing their ideas, see refs. 2 and 3.

5. CLUSTER EXPANSION OF BALANCED CONTOURS

In this section we perform the zeroth step of the multi-scale analysis for the polymer model (19), and set up the cluster expansion for a class of contours the weight of which is sufficiently damped. As a result, an interacting polymer model is obtained that will be dealt with in the next section.

Let an integer l_0 be fixed. It is supposed to be large enough and the precise conditions will be specified throughout the sequel. It plays the role of an η-independent "cut-off scale". Given any boundary condition η (fixed throughout this section), we start by defining the set of contours that allow for the cluster expansion. Obviously, every bulk contour Γ has the weight $\rho^\eta(\Gamma)=\exp(-2\beta|\Gamma|)$. For boundary contours, there is no such exponential bound with a strictly positive rate, uniformly in η. Instead, we segregate an η-dependent subset of sufficiently damped boundary contours as follows.

Definition 5.1. Given $\eta\in\Omega$, a boundary contour Γ is called balanced (or η-balanced) whenever

$$\sum_{x\in\underline{\partial^-\Gamma}} \eta_x \geqslant -\left(1-\frac{1}{l_0}\right)|\Gamma| \tag{23}$$

Otherwise Γ is called unbalanced.

A set $B\subset\partial\Lambda$ is called unbalanced if there exists an unbalanced contour Γ, $\partial^-\Gamma=B$.

While the case of large boundary contours will be discussed separately in the next section, some basic properties of unbalanced small boundary contours are collected in the following lemma. We define the *height* of any simple boundary contour Γ as

$$h(\Gamma) = \max_{y^* \in \Gamma} d[y^*, \partial\Gamma] \tag{24}$$

In order to extend this definition to small boundary contours Γ such that $\partial\Gamma$ contains an (exactly one) corner, we make the following construction. If $\partial\Gamma$ is a connected subset of the boundary with the endpoints $[\pm(N+1/2), a]$ and $[b, \pm(N+1/2)]$, then we define the set $R(\Gamma) \subset (\mathbb{Z}^2)^*$ as the (unique) rectangle such that $[\pm(N+1/2), a]$, $[b, \pm(N+1/2)]$, and $[\pm(N+1/2), \pm(N+1/2)]$ are three of its corners. Now the height is the maximal distance of a point in the contour to this rectangle,

$$h(\Gamma) = \max_{y^* \in \Gamma} d[y^*, R(\Gamma)] \tag{25}$$

The situation is illustrated in Fig. 6.

Lemma 5.2. Let Γ be an unbalanced small boundary contour. Then,

(i) $\sum_{x \in \underline{\partial\Gamma}} \eta_x \leqslant -\left(1 - \frac{2}{l_0}\right) |\partial\Gamma|$.

(ii) $|\partial\Gamma| \geqslant l_0 h(\Gamma)$. In particular, if Γ is simple then $|\partial\Gamma| \geqslant l_0$.

Proof. For any unbalanced contour Γ, Definition 5.1 together with the bound $|\Gamma| \geqslant |\partial\Gamma|$ valid for any small boundary contour implies the inequalities

$$-|\partial^-\Gamma| \leqslant \sum_{x \in \underline{\partial^-\Gamma}} \eta_x < -\left(1 - \frac{1}{l_0}\right) |\Gamma| \leqslant -\left(1 - \frac{1}{l_0}\right) |\partial\Gamma| \tag{26}$$

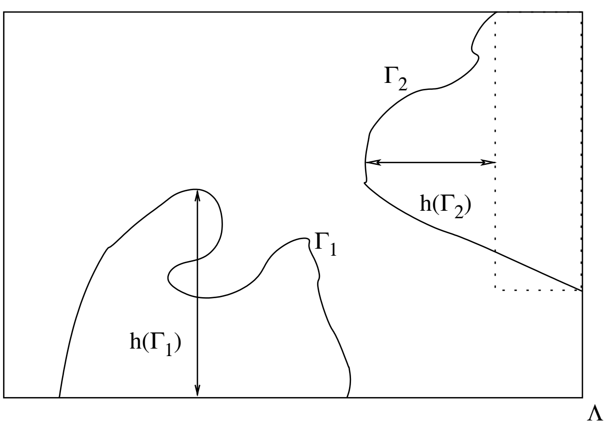

Fig. 6. Height of small boundary contours.

Hence, $|\partial^+\Gamma| \leqslant \frac{1}{l_0}|\partial\Gamma|$ and we obtain

$$\sum_{x\in\underline{\partial\Gamma}} \eta_x \leqslant -\left(1-\frac{1}{l_0}\right)|\partial\Gamma| + |\partial^+\Gamma| \leqslant -\left(1-\frac{2}{l_0}\right)|\partial\Gamma| \tag{27}$$

proving (i).

If Γ is simple, then we use (26) again together with the refined relation $|\Gamma| \geqslant |\partial\Gamma| + 2h(\Gamma)$ to get

$$|\partial\Gamma| \geqslant \left(1-\frac{1}{l_0}\right)|\Gamma| \geqslant \left(1-\frac{1}{l_0}\right)(|\partial\Gamma| + 2h(\Gamma)) \tag{28}$$

which implies $|\partial\Gamma| \geqslant l_0 h(\Gamma) \geqslant l_0$, assuming $l_0 \geqslant 2$ and using that $h(\Gamma) \geqslant 1$ for any simple small boundary contour.

Since the definition of the height is such that the inequality $|\Gamma| \geqslant |\partial\Gamma| + 2h(\Gamma)$ remains true as well for any small boundary contour Γ such that $\partial\Gamma$ contains a corner, the lemma is proven. ▮

The union of the set of all bulk contours and of the set of all balanced boundary contours is denoted by $\mathcal{K}_0^\eta$. We also write $\mathcal{D}_0^\eta$ for the set of all compatible families of contours from $\mathcal{K}_0^\eta$, and $\mathcal{D}_{>0}^\eta$ for the set of all compatible families of contours from $\mathcal{K}_\Lambda \setminus \mathcal{K}_0^\eta$. Later we will show that, for almost every η, all large boundary contours (i.e. those containing at least one interface) are balanced for all but finitely many squares $\Lambda(N)$.

Formally, the partition function (19) can be partially computed by summing over all contours from the set $\mathcal{K}_0^\eta$. We start by rewriting partition function (19) as

$$\mathcal{Z}_\Lambda^\eta = \exp(-E^\eta(\emptyset)) \sum_{\partial\in\mathcal{D}_{>0}^\eta} \prod_{\Gamma\in\partial} \rho^\eta(\Gamma) \sum_{\substack{\partial^0\in\mathcal{D}_0^\eta \\ \partial^0\sim\partial}} \prod_{\Gamma^0\in\partial^0} \rho^\eta(\Gamma^0) \tag{29}$$

Here, the first sum runs over all compatible families ∂ of contours not belonging to $\mathcal{K}_0^\eta$, while the second one is over all collections of contours from $\mathcal{K}_0^\eta$, compatible with ∂. Let $\mathfrak{C}_\Lambda^0$ denote the set of all clusters of contours from $\mathcal{K}_0^\eta$. Then, the cluster expansion reads, see Appendix A,

$$\mathcal{Z}_\Lambda^\eta = \exp(-E^\eta(\emptyset)) \sum_{\partial\in\mathcal{D}_{>0}^\eta} \prod_{\Gamma\in\partial} \rho^\eta(\Gamma) \exp\left(\sum_{\substack{C\in\mathfrak{C}_0^\eta \\ C\sim\partial}} \phi_0^\eta(C)\right) \tag{30}$$

where the sum runs over all clusters of contours from $\mathcal{K}_0^\eta$ that are compatible with ∂, and we have denoted the weight of a cluster C by $\phi_0^\eta(C)$. Note that the cluster expansion was applied only formally here and it needs to be justified by providing bounds on the cluster weights. This is done in Proposition 5.4 below.

Hence, we rewrite the model with the partition function $\mathcal{Z}_\Lambda^\eta$ as an effective model upon the contour ensemble $\mathcal{K}_\Lambda \setminus \mathcal{K}_0^\eta$, with a contour interaction mediated by the clusters:

$$\mathcal{Z}_\Lambda^\eta = \mathcal{Z}_1^\eta \exp\left(-E^\eta(\emptyset) + \sum_{C\in\mathfrak{C}_0^\eta} \phi_0^\eta(C)\right) \tag{31}$$

where

$$\mathcal{Z}_1^\eta = \sum_{\partial\in\mathcal{D}_{>0}^\eta} \exp\left(-\sum_{\substack{C\in\mathfrak{C}_0^\eta\\ C\not\sim\partial}} \phi_0^\eta(C)\right)\prod_{\Gamma\in\partial}\rho^\eta(\Gamma) \tag{32}$$

After establishing an exponential upper bound on the number of incompatible contours in the next lemma, a bound on the cluster weights immediately follows by recalling the basic result on the convergence of the cluster expansions.[29]

Lemma 5.3. There exists a constant $c_1>0$ (independent of l_0) such that the number of all contours $\Gamma'\in\mathcal{K}_\Lambda$, $|\Gamma'|=n$, $\Gamma'\not\sim\Gamma$ is upper-bounded by $|\Gamma|\,e^{c_1 n}$, for any $\Gamma\in\mathcal{K}_\Lambda$ and $n=1,2,\dots$

Proof. Note that Γ is not necessarily a connected set. However, the relation $\Gamma'\not\sim\Gamma$ implies $(\Gamma'\cup\partial\Gamma')\cap(\Gamma\cup\partial\Gamma)\neq\emptyset$, and using that $\Gamma\cup\partial\Gamma$ is connected, we get:

$$\begin{aligned}
\#\{\Gamma' : \Gamma'\not\sim\Gamma,\ |\Gamma'|=n\} &\leqslant |\Gamma\cup\partial\Gamma| \sup_{x^*}\#\{\Gamma' : x^*\in\Gamma'\cup\partial\Gamma',\ |\Gamma'|=n\}\\
&\leqslant 3|\Gamma| \sup_{x^*}\#\{A\subset(\mathbb{Z}^2)^* \text{ connected},\ x^*\in A,\ |A|\leqslant 3n\}\\
&\leqslant |\Gamma|\cdot 4^{6n+1}\leqslant |\Gamma|\,e^{c_1 n}
\end{aligned}$$

by choosing c_1 large enough. ∎

Assigning to any cluster $C\in\mathfrak{C}_0^\eta$ the *domain* $\mathrm{Dom}(C)=\underline{\partial C}$ where $\partial C=\cup_{\Gamma\in C}\partial\Gamma$ is the boundary of C, and the length $|C|=\sum_{\Gamma\in C}|\Gamma|$, we have the following result.

Proposition 5.4. There are constants $\beta_4, c_2>0$ (independent of l_0) such that for any $\beta\geqslant l_0\beta_4$, one has the upper bound

$$\sup_{x^*} \sum_{\substack{C\in\mathfrak{C}_0^\eta \\ x^*\in C}} |\phi_0^\eta| \exp\left[\left(\frac{2\beta}{l_0} - c_2\right)|C|\right] \leqslant 1 \tag{33}$$

uniformly in Λ.

Moreover, $\phi_0^\eta(C)$ only depends on the restriction of η to the set $\mathrm{Dom}(C)$.

Proof. Using Definition 5.1 and Eq. (21), we have $\rho^\eta(\Gamma) \leqslant \exp(-\frac{2\beta}{l_0}|\Gamma|)$ for any balanced contour Γ. In combination with Lemma 5.3, we get

$$\sum_{\substack{\Gamma\in\mathcal{K}_0^\eta \\ x^*\in\Gamma}} |\rho^\eta(\Gamma)| \exp\left[\left(\frac{2\beta}{l_0} - c_2 + 1\right)|\Gamma|\right] \leqslant \sum_{n=1}^{\infty} \exp[-(c_2 - c_1 - 1)n] \leqslant 1 \tag{34}$$

provided that c_2 is chosen large enough. The proposition now follows by applying Proposition A.2, with $\beta_4 = \frac{c_2}{2}$. ∎

6. ABSENCE OF LARGE BOUNDARY CONTOURS

By the construction, all unbalanced contours are boundary contours, either small or large. In this section we show that unbalanced large boundary contours actually do not exist under a typical realization of the boundary condition. This observation will allow us to restrict our multiscale analysis entirely to the class of small boundary contours.

Lemma 6.1. There is a constant $c_3 > 0$ such that for any $N \in \mathbb{N}$ and any unbalanced large boundary contour $\Gamma \in \mathcal{K}_{\Lambda(N)}$, the inequality

$$\sum_{x\in\underline{\partial\Gamma}} \eta_x \leqslant -c_3 N \tag{35}$$

holds true.

Proof. Using the geometrical inequality $|\Gamma| \geqslant 2N + |\partial^+\Gamma|$ and Definition 5.1, we have

$$\begin{aligned}\sum_{x\in\underline{\partial\Gamma}} \eta_x &\leqslant -\left(1 - \frac{1}{l_0}\right)|\Gamma| + \sum_{x\in\underline{\partial^+\Gamma}} \eta_x \\ &\leqslant -\left(1 - \frac{1}{l_0}\right)(2N + |\partial^+\Gamma|) + |\partial^+\Gamma| \\ &\leqslant -2N\left(1 - \frac{3}{l_0}\right)\end{aligned} \tag{36}$$

where in the last inequality we used that $|\partial^+\Gamma| \leqslant |\partial\Gamma| \leqslant 4N$. ∎

Proposition 6.2. There is a constant $c_4 > 0$ such that for any $N \in \mathbb{N}$,

$$\boldsymbol{P}\{\exists \Gamma \in \mathcal{K}_{\Lambda(N)} \text{ large unbalanced}\} \leqslant \exp(-c_4 N) \tag{37}$$

Proof. If $B \subset \partial\Lambda(N)$ is a connected set containing exactly two corners, then, using Lemma 6.1,

$$\begin{aligned} &\boldsymbol{P}\{\exists \Gamma \in \mathcal{K}_{\Lambda(N)} \text{ large unbalanced} : \partial\Gamma = B\} \leqslant \boldsymbol{P}\left\{\textstyle\sum_{x \in \underline{B}} \eta_x \leqslant -c_3 N\right\} \\ &\quad \leqslant \boldsymbol{P}\left\{\textstyle\sum_{x \in \underline{B}} \eta_x \leqslant -\frac{c_3}{2}|B|\right\} \leqslant \exp\left(-\frac{c_3^2}{8}|B|\right) \end{aligned} \tag{38}$$

Hence,

$$\begin{aligned} &\boldsymbol{P}\{\exists \Gamma \in \mathcal{K}_{\Lambda(N)} \text{ large unbalanced}\} \\ &\quad \leqslant \sum_{B \subset \partial\Lambda} \boldsymbol{P}\{\exists \Gamma \in \mathcal{K}_{\Lambda(N)} \text{ large unbalanced} : \partial\Gamma = B\} \\ &\quad \leqslant \sum_{l \geqslant 2N} 8N \exp\left(-\frac{c_3^2}{8} l\right) \leqslant \frac{128N}{c_3^2} \exp\left(-\frac{c_3^2 N}{4}\right) \leqslant \exp(-c_4 N) \end{aligned} \tag{39}$$

for N large enough and an appropriate c_4. Choosing c_4 small enough gives (37) for all N. ∎

Corollary 6.3. There exist a set $\Omega^* \subset \Omega$, $\boldsymbol{P}\{\Omega^*\} = 1$ and a function $N^* : \Omega^* \mapsto \mathbb{N}$ such that for any b.c. $\eta \in \Omega^*$ and any volume $\Lambda = \Lambda(N)$, $N \geqslant N^*(\eta)$, all large boundary contours are balanced.

Proof. Since

$$\sum_N \boldsymbol{P}\{\exists \Gamma \in \mathcal{K}_{\Lambda(N)} \text{ large unbalanced}\} < \infty \tag{40}$$

the Borel–Cantelli lemma implies

$$\boldsymbol{P}\{\forall N_0 \in \mathbb{N} : \exists N \geqslant N_0 : \ \exists \Gamma \in \mathcal{K}_{\Lambda(N)} \text{ large unbalanced}\} = 0 \tag{41}$$

proving the statement. ∎

We are now ready to prove the almost sure absence of interfaces in the large-volume limit.

Proof of Theorem 4.1. Let $\eta \in \Omega^* \cap (-\Omega^*)$ and $N \geqslant N^*(\eta)$. Then, any large boundary contour Γ is both η- and $(-\eta)$-balanced and, using the Peierls inequality and (15), the Gibbs probability of any collection of (possibly large boundary) contours $\Gamma_1, \ldots, \Gamma_m$, $m = 1, 2, \ldots$ has the upper bound

$$\begin{aligned}\mu^{\eta}_{\Lambda(N)}(\Gamma_1, \ldots, \Gamma_m) &\leqslant \max_{a \in \{-1,1\}} \nu^{a\eta}_{\Lambda(N)}(\Gamma_1, \ldots, \Gamma_m) \\ &\leqslant \max_{a \in \{-1,1\}} \prod_{i=1}^{m} \rho^{a\eta}(\Gamma_i) \leqslant \exp\left(-\frac{2\beta}{l_0}\sum_{i=1}^{m}|\Gamma_i|\right)\end{aligned} \tag{42}$$

Hence, using Lemma 5.3 and the bound $|\Gamma| \geqslant 2N$ for any large boundary contour Γ, we get

$$\begin{aligned}\mu^{\eta}_{\Lambda(N)}(\exists \text{ a large boundary contour}) &\leqslant \sum_{m=1}^{\infty}\frac{1}{m!}\sum_{\substack{\Gamma_1,\ldots,\Gamma_m \text{ large} \\ \forall i:\Gamma_i\cap\partial\Lambda\neq\emptyset}} \mu^{\eta}_{\Lambda(N)}(\Gamma_1, \ldots, \Gamma_m) \\ &\leqslant \sum_{m=1}^{\infty}\frac{1}{m!}\sum_{x_1,\ldots,x_m\in\partial\Lambda}\;\sum_{\Gamma_1\ni x_1,\ldots,\Gamma_m\ni x_m}\exp\left(-\frac{2\beta}{l_0}\sum_{i=1}^{m}|\Gamma_i|\right) \\ &\leqslant \exp\left(-\frac{2\beta}{l_0}N\right)\sum_{m=1}^{\infty}\frac{1}{m!}\left(4N\sum_{\substack{\Gamma\ni x \\ |\Gamma|\geqslant 2N}} e^{-\frac{\beta}{l_0}|\Gamma|}\right)^{m} \\ &\leqslant \exp\left[-\left(\frac{2\beta}{l_0} - 8e^{-2\left(\frac{\beta}{l_0}-c_1\right)N}\right)N\right] \longrightarrow 0\end{aligned} \tag{43}$$

provided that β is large enough. Since $\boldsymbol{P}\{\Omega^* \cap (-\Omega^*)\} = 1$, the theorem is proven. ∎

As a consequence, all interfaces get $\boldsymbol{P}$-a.s. and for all but finitely many volumes uniformly exponentially damped weights. Hence, their Gibbs probabilities become exponentially small as functions of the size of the system and, therefore, no interfacial infinite-volume Gibbs measure occurs as a limit point, with $\boldsymbol{P}$-probability 1. While such a result is not sensational in $d = 2$ (in this case, no translationally non-invariant Gibbs measure exists by refs. 1 and 26), similar arguments are expected to apply in higher dimensions.

In the next sections, a perturbation technique is developed that allows us to address the question whether non-trivial mixtures of μ^+ and μ^- can occur as limit measures.

7. CLASSIFICATION OF UNBALANCED CONTOURS

We now consider the interacting contour model introduced by the partition function (32), defined on the set of unbalanced contours $\mathcal{K}_\Lambda \setminus \mathcal{K}_0^\eta$. As a consequence of Corollary 6.3, we can restrict our analysis to the set Ω^* of boundary conditions under which the set $\mathcal{K}_\Lambda \setminus \mathcal{K}_0^\eta$ of unbalanced contours contains only small boundary contours, both simple and corner ones.

Our multi-scale analysis consists of a sequential expansion of groups of unbalanced contours that are far enough from each other. The groups are supposed to be typically sufficiently rarely distributed, so that the partition function (31) can be expanded around the product over the partition functions computed within these groups only. Under the condition that the density of the groups decays fast enough with their space extension, one can arrive at an expansion that essentially shares the locality features of the usual cluster expansion, at least for $\boldsymbol{P}$-typical boundary conditions η. To make this strategy work, we define a suitable decomposition of the set $\mathcal{K}_\Lambda \setminus \mathcal{K}_0^\eta$ into disjoint groups associated with a hierarchy of length scales. Also, the unbalanced contours close enough to any of the four corners will be dealt with differently and expanded in the end.

Definition 7.1. Assuming l_0 to be fixed, we define the two sequences $(l_n)_{n=1,2,\dots}$ and $(L_n)_{n=1,2,\dots}$ by the following recurrence relations:

$$L_n = \frac{l_{n-1}}{5^n}, \quad l_n = \exp\left(\frac{L_n}{2^n}\right) \quad n = 1, 2, \dots \tag{44}$$

For any $n = 1, 2, \dots$, any pair of contours Γ, Γ' is called L_n-connected, if $d[\Gamma, \Gamma'] \leqslant L_n$. Furthermore, fixing a positive constant $\epsilon > 0$, we introduce the N-dependent length scale

$$l_\infty = (\log N)^{1+\epsilon} \tag{45}$$

Introducing the boundary $\partial\Delta$ for any set of contours $\Delta \subset \mathcal{K}_\Lambda$ by $\partial\Delta = \cup_{\Gamma\in\Delta} \partial\Gamma$, we consider the η-dependent decomposition of the set of contours $\mathcal{K}_\Lambda \setminus \mathcal{K}_0^\eta$ defined by induction as follows.

Definition 7.2.

(1) A maximal L_1-connected subset $\Delta \subset \mathcal{K}_\Lambda \setminus \mathcal{K}_0^\eta$ is called a 1-*aggregate* whenever (i) $|\partial\Delta|_{\mathrm{con}} \leqslant l_1$, (ii) there is no corner $x^*_{C,i}$ such that $\max_{y^*\in\partial\Delta} d[y^*, x^*_{C,i}] \leqslant l_\infty$. We use the notation $(\mathcal{K}^\eta_{1,\alpha})$ for the collection of all 1-aggregates, and write $\mathcal{K}_1^\eta = \cup_\alpha \mathcal{K}^\eta_{1,\alpha}$.

$\vdots$

(n) Assume the sets $(\mathcal{K}^{\eta}_{j,\alpha})_{j=1,\dots,n-1}$ have been defined. Then, the n-aggregates are defined as maximal L_n-connected subsets $\Delta \subset \mathcal{K}_\Lambda \setminus \cup_{j<n}\mathcal{K}^{\eta}_j$ satisfying (i) $|\partial\Delta|_{\mathrm{con}} \leqslant l_n$, (ii) there is no corner $x^*_{C,i}$ such that $\max_{y^*\in\partial\Delta} d[y^*, x^*_{C,i}] \leqslant l_\infty$. The set of all n-aggregates is denoted by $(\mathcal{K}^{\eta}_{n,\alpha})$, and $\mathcal{K}^{\eta}_n = \cup_\alpha \mathcal{K}^{\eta}_{n,\alpha}$.

To each n-aggregate $\mathcal{K}^{\eta}_{n,\alpha}$ we assign the domain

$$\mathrm{Dom}(\mathcal{K}^{\eta}_{n,\alpha}) := \underline{\{x^* \in \partial\Lambda;\ d[x^*, \partial\mathcal{K}^{\eta}_{n,\alpha}] \leqslant L_n\}} \tag{46}$$

Obviously, the set $\mathcal{K}^{\eta}_\infty := \mathcal{K}_\Lambda \setminus (\mathcal{K}^{\eta}_1 \cup \mathcal{K}^{\eta}_2 \cup \cdots)$ need not be empty, and since all large boundary contours are balanced, for every contour $\Gamma \in \mathcal{K}^{\eta}_\infty$ there is exactly one corner $x^*_{C,i}$ such that $\max_{y^*\in\partial\Gamma} d[y^*, x^*_{C,i}] \leqslant l_\infty$. Hence, there is a natural decomposition of the set $\mathcal{K}^{\eta}_\infty$ into at most four *corner aggregates*, $\mathcal{K}^{\eta}_\infty = \cup_i \mathcal{K}^{\eta}_{\infty,i}$, each of them consisting of contours within the logarithmic neighborhood of one of the corners. In general, any corner aggregate contains both simple and corner boundary contours. Later we will show that with $\boldsymbol{P}$-probability 1, every unbalanced corner boundary contour belongs to a corner aggregate. In other words, every n-aggregate, $n = 1, 2, \dots$ contains only simple boundary contours.

Remark 7.3. By Definition 7.2, any n-aggregate has a distance at least L_n from all m-aggregates, $m \geqslant n$. In this way, in the nth step of our expansion, after having removed all lower-order aggregates, we will be able to use the "essential independence" of all n-aggregates. Namely, on the assumption that L_n is big enough, depending on the aggregate size l_n, both the interaction among the n-aggregates and the interaction between n-aggregates and m-aggregates, $m \geqslant n$ will be controlled by a cluster expansion.

Our first observation is a local property of the above construction, which will be crucial to keep the dependence of expansion terms to be defined later depending only on a sufficiently small set of boundary spins.

Lemma 7.4. Let a set of small boundary contours Δ be fixed and assume that $\eta, \eta' \in \Omega$ are such that $\eta_{\mathrm{Dom}(\Delta)} = \eta'_{\mathrm{Dom}(\Delta)}$. Then, Δ is an n-aggregate w.r.t. the boundary condition η if and only if it is an n-aggregate w.r.t. η'.

The super-exponential growth of the scales l_n will imply an exponential decay of the probability for an n-aggregate to occur. An upper bound on this probability is stated in the following proposition, the proof of which is given in Section 11.1.

Proposition 7.5. There is a constant $c_5>0$ (independent of l_0) such that for any $n=1,2,\dots$ and any connected set $B\subset\partial\Lambda$,

$$\boldsymbol{P}\{\exists\mathcal{K}^\eta_{n,\alpha}:\mathrm{Con}(\partial\mathcal{K}^\eta_{n,\alpha})=B\}\leqslant e^{-c_5|B|} \tag{47}$$

uniformly in Λ.

Note that, given a connected set $B\subset\partial\Lambda$, there is at most one aggregate $\mathcal{K}^\eta_{n,\alpha}$, $n=1,2,\dots$ such that $\mathrm{Con}\cdot(\partial\mathcal{K}^\eta_{n,\alpha})=B$.

Corollary 7.6. There exists $\Omega^{**}\subset\Omega^*$, $\boldsymbol{P}\{\Omega^{**}\}=1$ and $N^{**}:\omega^{**}\mapsto\mathbb{N}$, $N^{**}(\omega)\geqslant N^*(\omega)$ such that for any $\omega\in\Omega^{**}$ and any $\Lambda=\Lambda(N)$, $N\geqslant N^{**}(\omega)$ every aggregate $\mathcal{K}^\eta_{n,\alpha}$, $n=1,2,\dots$ satisfies the inequality $|\partial\mathcal{K}^\eta_{n,\alpha}|_{\mathrm{con}}\leqslant l_\infty$. In particular:

(i) The set $\mathrm{Con}(\mathcal{K}^\eta_{n,\alpha})$ is a boundary interval and there is at most one corner $x^*_{C,i}$ such that $d[x^*_{C,i},\partial\mathcal{K}^\eta_{n,\alpha}]\leqslant l_\infty$.

(ii) All contours $\Gamma\in\mathcal{K}^\eta_{n,\alpha}$ are simple boundary contours.

Proof. Using Proposition 7.5, the probability for any aggregate to occur can be estimated as

$$\boldsymbol{P}\{\exists\mathcal{K}^\eta_{n,\alpha},n=1,2,\dots:|\partial\mathcal{K}^\eta_{n,\alpha}|_{\mathrm{con}}>l_\infty\}\leqslant\sum_{\substack{B\subset\partial\Lambda\ \text{conn.}\\|B|>(\log N)^{1+\epsilon}}}\boldsymbol{P}\{\exists\mathcal{K}^\eta_{n,\alpha}:\mathrm{Con}(\mathcal{K}^\eta_{n,\alpha})=B\}$$
$$\leqslant|\partial\Lambda|\sum_{l>(\log N)^{1+\epsilon}}e^{-c_5 l}\leqslant\frac{16}{c_5}N^{1-c_5(\log N)^\epsilon}=o\left(N^{-\delta}\right) \tag{48}$$

for any (arbitrarily large) $\delta>0$. Hence,

$$\sum_{N=1}^{\infty}\boldsymbol{P}\{\exists\mathcal{K}^\eta_{n,\alpha},\ n=1,2,\dots:\ |\partial\mathcal{K}^\eta_{n,\alpha}|_{\mathrm{con}}>l_\infty\}<\infty \tag{49}$$

and the statement follows by a Borel–Cantelli argument. ▮

For convenience, let us summarize the results of the last three sections by reviewing all types of contours again together with their balancedness properties. For any $\eta\in\Omega^{**}$ and $\Lambda=\Lambda(N)$, $N\geqslant N^{**}(\omega)$, any configuration of contours $\partial\in\mathcal{D}_\Lambda$ possibly contains

(i) *Bulk contours* (trivially balanced).

(ii) *Large boundary contours* that are balanced.

(iii) *Corner boundary contours* that are either balanced or elements of corner aggregates.

(iv) *Simple boundary contours* which are balanced or elements of either n-aggregates, $n=1,2,\dots,$ or of corner aggregates.

8. SEQUENTIAL EXPANSION OF UNBALANCED CONTOURS

The next step in our strategy is to proceed by induction in the order of aggregates, rewriting at each step the interacting polymer model (32) as an effective model over the contour ensembles $\mathcal{K}_\Lambda \setminus (\mathcal{K}_0^\eta \cup \mathcal{K}_1^\eta)$, $\mathcal{K}_\Lambda \setminus (\mathcal{K}_0^\eta \cup \mathcal{K}_1^\eta \cup \mathcal{K}_2^\eta)$, etc. At the nth step, a compatible set of contours inside all corner and all normal m-aggregates, $m>n$, is fixed, and we perform the summation over contours in all normal n-aggregates. This is a constrained partition function which is approximately a product over the normal n-aggregates. By the construction, the latter are sufficiently isolated on the scale L_n, which will allow for the control of the remaining interaction by means of a cluster expansion. At the end, we arrive at an effective model over the contour ensemble $\mathcal{K}_\infty^\eta$, which is the union of (at most four) corner aggregates. In large volumes, the corner aggregates become essentially independent, the error being exponentially small in the size of the volume. The reason we distinguish between the n-aggregates and the corner aggregates is that the partition function within the former allows for a much better control, which will be essential in our analysis of the characteristic function of the random free energy difference $\log \mathcal{Z}_\Lambda^\eta - \log \mathcal{Z}_\Lambda^{-\eta}$ in Section 10. Note that the lack of detailed control around the corners is to be expected as there may more easily occur some low-energy (unbalanced) boundary contours, but at most of logarithmic size in N.

The nth step of the expansion, $n \geqslant 1$, starts from the partition function,

$$\mathcal{Z}_n^\eta = \sum_{\partial \in \mathcal{D}_{>n-1}^\eta} \exp\left(- \sum_{\substack{C \in \mathfrak{C}_{n-1}^\eta \\ C \nsim \partial}} \phi_{n-1}^\eta(C) \right) \prod_{\Gamma \in \partial} \rho^\eta(\Gamma) \tag{50}$$

which in the case $n=1$ coincides with (32). Here, $\mathcal{D}_{>n-1}^\eta$ is the set of all compatible families of contours from $\mathcal{K}_{>n-1}^\eta := \mathcal{K}_\Lambda^\eta \setminus (\mathcal{K}_0^\eta \cup \mathcal{K}_1^\eta \cup \cdots \cup \mathcal{K}_{n-1}^\eta)$, i.e. with all normal m-aggregates, $m \leqslant n-1$, being removed. Furthermore, we use the notation $\mathfrak{C}_{n-1}^\eta$ for the set of all $(n-1)$-clusters. Here, the 0-clusters have been introduced in Section 5, and the clusters of higher order will be defined inductively in the sequel.

In order to analyze partition function (50), we follow the ideas of Fröhlich and Imbrie[21], however, we choose to present them in a slightly different way. Observing that, by construction, the family of aggregates compose a "sparse set", one is tempted to approximate the partition function by a product over the aggregates and to control the error by means of a cluster expansion. However, to make this strategy work, we need to "renormalize" suitably the contour weights. Namely, only the clusters that intersect at least two distinct aggregates generate an interaction between them, and are sufficiently damped by using the sparsity of the set of aggregates. On the other hand, the (sufficiently short) clusters intersecting a single aggregate cannot be expanded, and they modify the weights of contour configurations within the aggregate. An important feature of this procedure is that the weight of these contour configurations is kept positive. In some sense, it is this very renormalization of the weights within each aggregate that can hardly be done via a single expansion and requires an inductive approach. In what follows, we present this strategy in detail, via a number of steps.

8.1. Renormalization of Contour Weights

For any compatible set of contours $\partial \subset \mathcal{K}_n^\eta$, define the renormalized weight

$$\hat{\rho}^\eta(\partial) = \exp\left(- \sum_{\substack{C \in \mathfrak{C}_{n-1}^\eta \\ C \nsim \partial;\, |C| < L_n}} \phi_{n-1}^\eta(C) \right) \prod_{\Gamma \in \partial} \rho^\eta(\Gamma) \tag{51}$$

Note that the above sum only includes the clusters of length smaller than L_n. By construction, any such cluster is incompatible with at most one n-aggregate. Hence, the renormalized weight $\hat{\rho}^\eta(\Gamma)$ factorizes over the n-aggregates and we have $\hat{\rho}^\eta(\partial) = \prod_\alpha \hat{\rho}^\eta(\partial^\alpha)$ where $\partial^\alpha = \partial \cap \mathcal{K}_{n,\alpha}^\eta$. Therefore, formula (50) gets the form

$$\begin{aligned}
\mathcal{Z}_n^\eta &= \sum_{\partial \in \mathcal{D}_{>n}^\eta} \prod_{\Gamma \in \partial} \rho^\eta(\Gamma) \sum_{\partial^n \in \mathcal{D}_n^\eta} \hat{\rho}^\eta(\partial^n) \exp\left(- \sum_{\substack{C \in \mathfrak{C}_{n-1}^\eta \\ (C \nsim \partial) \vee (C \nsim \partial^n;\, |C| \geqslant L_n)}} \phi_{n-1}^\eta(C) \right) \\
&= \sum_{\partial \in \mathcal{D}_{>n}} \prod_{\Gamma \in \partial} \rho^\eta(\Gamma) \exp\left(- \sum_{\substack{C \in \mathfrak{C}_{n-1}^\eta \\ C \nsim \partial}} \phi_{n-1}^\eta(C) \right) \\
&\quad \times \sum_{\partial^n \in \mathcal{D}_n^\eta} \hat{\rho}^\eta(\partial^n) \exp\left(- \sum_{\substack{C \in \mathfrak{C}_{n-1}^\eta;\, |C| \geqslant L_n \\ C \sim \partial;\, C \nsim \partial^n}} \phi_{n-1}^\eta(C) \right)
\end{aligned} \tag{52}$$

Defining the renormalized partition function $\hat{\mathcal{Z}}^{\eta}_{n,\alpha}$ of the contour ensemble $\mathcal{K}^{n,\alpha}_{\Lambda}$ as

$$\hat{\mathcal{Z}}^{\eta}_{n,\alpha} = \sum_{\partial^n \in \mathcal{D}^{\eta}_{n,\alpha}} \hat{\rho}^{\eta}(\partial^n) \tag{53}$$

and using the shorthand

$$\tilde{\phi}^{\eta}_{n-1}(C, \partial^n) = \phi^{\eta}_{n-1}(C)\mathbf{1}_{\{C \nsim \partial^n;\, |C| \geqslant L_n\}} \tag{54}$$

we obtain

$$\begin{aligned}\mathcal{Z}^{\eta}_{n} = \prod_{\alpha} \hat{\mathcal{Z}}^{\eta}_{n,\alpha} \sum_{\partial \in \mathcal{D}^{\eta}_{>n}} \prod_{\Gamma \in \partial} \rho^{\eta}(\Gamma) \exp\left(- \sum_{\substack{C \in \mathfrak{C}^{\eta}_{n-1} \\ C \nsim \partial}} \phi^{\eta}_{n-1}(C) \right) \\ \times \sum_{\partial^n \in \mathcal{D}^{\eta}_{n}} \prod_{\alpha} \frac{\hat{\rho}^{\eta}(\partial^{n,\alpha})}{\hat{\mathcal{Z}}^{\eta}_{n,\alpha}} \exp\left(- \sum_{\substack{C \in \mathfrak{C}^{\eta}_{n-1} \\ C \sim \partial}} \tilde{\phi}^{\eta}_{n-1}(C, \partial^n) \right)\end{aligned} \tag{55}$$

where $\partial^{n,\alpha} = \partial^n \cap \mathcal{K}^{\eta}_{n,\alpha}$ is the restriction of ∂^n to the n-aggregate $\mathcal{K}^{\eta}_{n,\alpha}$. In the last expression, the second sum contains the interaction between n-aggregates, to make a correction to the product over the renormalized partition functions $\hat{\mathcal{Z}}^{\eta}_{n,\alpha}$.

8.2. Cluster Expansion of the Interaction Between *n*-Aggregates

Now we employ a trick familiar from the theory of high-temperature (Mayer) expansions, and assign to any family $\mathcal{C} \subset \mathfrak{C}^{\eta}_{n-1}$ of $(n-1)$-clusters the weight

$$w^{\eta}_{n}(\mathcal{C}) = \frac{1}{\prod_{\alpha} \hat{\mathcal{Z}}^{\eta}_{n,\alpha}} \sum_{\partial^n \in \mathcal{D}^{\eta}_{n}} \hat{\rho}^{\eta}(\partial^n) \prod_{C \in \mathcal{C}} \left(e^{-\tilde{\phi}^{\eta}_{n-1}(C, \partial^n)} - 1 \right) \tag{56}$$

See Fig. 7 for an example of a family of 1-clusters that generically gets a non-trivial weight according to this construction.

Definition 8.1. Any pair of $(n-1)$-clusters $C_1, C_2 \in \mathfrak{C}^{\eta}_{n-1}$ is called *n-incompatible*, $C_1 \overset{n}{\nleftrightarrow} C_2$, whenever there exists an n-aggregate $\mathcal{K}^{\eta}_{n,\alpha}$ such that $C_1 \nsim \mathcal{K}^{\eta}_{n,\alpha}$ and $C_2 \nsim \mathcal{K}^{\eta}_{n,\alpha}$. In general, the sets $\mathcal{C}_1, \mathcal{C}_2 \subset \mathfrak{C}^{\eta}_{n-1}$ are *n-incompatible* if there are $C_1 \in \mathcal{C}_1$, $C_2 \in \mathcal{C}_2$, $C_1 \overset{n}{\nleftrightarrow} C_2$.

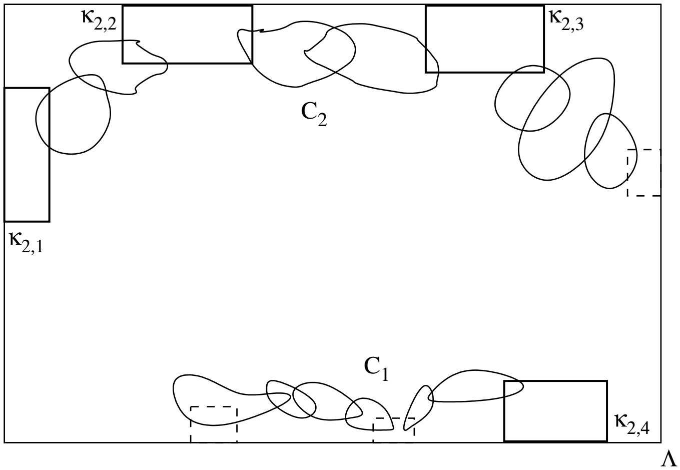

Fig. 7. A pair of 2-compatible families of 1-clusters $\mathcal{C}_1, \mathcal{C}_2 \subset \mathfrak{C}_1^\eta$ intersecting 2-aggregates $K_{2,\alpha}^\eta$, $\alpha = 1, 2, 3, 4$. The dashed rectangles illustrate 1-aggregates which have become parts of the 1-clusters. By construction, $w_2^\eta(\mathcal{C}_1 \cup \mathcal{C}_2) = w_2^\eta(\mathcal{C}_1)\, w_2^\eta(\mathcal{C}_2)$.

One easily checks the following properties of the weight $w_n^\eta(\mathcal{C})$.

Lemma 8.2. For any set of $(n-1)$-clusters $\mathcal{C} \in \mathfrak{C}_{n-1}^\eta$,

(i) $\sup_\eta |w_n^\eta(\mathcal{C})| \leqslant \prod_{C\in\mathcal{C}} (e^{|\phi_{n-1}^\eta(C)|} - 1)$.

(ii) If $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ such that $\mathcal{C}_1 \overset{n}{\leftrightarrow} \mathcal{C}_2$, then $w_n^\eta(\mathcal{C}) = w_n^\eta(\mathcal{C}_1)\, w_n^\eta(\mathcal{C}_2)$.

(iii) The weight $w_n^\eta(\mathcal{C})$ depends only on the restriction of η to the set $(\cup_{C\in\mathcal{C}} \mathrm{Dom}(C)) \cup (\cup'_\alpha \mathrm{Dom}(\mathcal{K}_{n,\alpha}^\eta))$ where the second union is over all n-aggregates $\mathcal{K}_{n,\alpha}^\eta$ such that $\mathcal{C} \nsim \mathcal{K}_{n,\alpha}^\eta$.

In the second sum in (55) we recognize the partition function of a polymer model with the polymers being defined as the n-connected subsets of $\mathfrak{C}_{n-1}^\eta$, which are incompatible if and only if they are n-incompatible. Treating this polymer model by the cluster expansion, and using the symbols $\mathfrak{D}_n^\eta$ for the set of all clusters in this polymer model and $\psi_n^\eta(D)$ for the weight of a cluster $D \in \mathfrak{D}_n^\eta$, we get

$$\mathcal{Z}_n^\eta = \prod_\alpha \hat{\mathcal{Z}}_{n,\alpha}^\eta \sum_{\partial\in\mathcal{D}_{>n}^\eta} \prod_{\Gamma\in\partial} \rho^\eta(\Gamma) \exp\left(- \sum_{\substack{C\in\mathfrak{C}_{n-1}^\eta \\ C\nsim\partial}} \phi_{n-1}^\eta(C) \right) \sum_{\substack{\mathcal{C}\subset\mathfrak{C}_{n-1}^\eta \\ \mathcal{C}\sim\partial}} w_n^\eta(\mathcal{C})$$

$$= \exp\left(\sum_{D\in\mathfrak{D}_n^\eta} \psi_n^\eta(D)\right) \prod_\alpha \hat{\mathcal{Z}}_{n,\alpha}^\eta \sum_{\partial\in\mathcal{D}_{>n}^\eta} \prod_{\Gamma\in\partial} \rho^\eta(\Gamma)$$
$$\times \exp\left(-\sum_{\substack{C\in\mathfrak{C}_{n-1}^\eta \\ C\nsim\partial}} \phi_{n-1}^\eta(C) - \sum_{\substack{D\in\mathfrak{D}_n^\eta \\ D\nsim\partial}} \psi_n^\eta(D)\right) \tag{57}$$

Defining the set of all n-clusters $\mathfrak{C}_n^\eta = \mathfrak{C}_{n-1}^\eta \cup \mathfrak{D}_n^\eta$ and the weight of any n-cluster $C \in \mathfrak{C}_n^\eta$ as

$$\phi_n^\eta(C) = \begin{cases} \phi_{n-1}^\eta(C) & \text{if } C \in \mathfrak{C}_{n-1}^\eta \\ \psi_n^\eta(C) & \text{if } C \in \mathfrak{D}_n^\eta \end{cases} \tag{58}$$

we finish the inductive step by obtaining the final expression

$$\mathcal{Z}_n^\eta = \mathcal{Z}_{n+1}^\eta \prod_\alpha \hat{\mathcal{Z}}_{n,\alpha}^\eta \exp\left(\sum_{D\in\mathfrak{D}_n^\eta} \psi_n^\eta(D)\right) \tag{59}$$

with the partition function of a new interacting polymer model

$$\mathcal{Z}_{n+1}^\eta = \sum_{\partial\in\mathrm{D}_{>n}^\eta} \exp\left(-\sum_{\substack{C\in\mathfrak{C}_n^\eta \\ C\nsim\partial}} \phi_n^\eta(C)\right) \prod_{\Gamma\in\partial} \rho^\eta(\Gamma) \tag{60}$$

We need to extend the notion of domain from the set of $(n-1)$-clusters $\mathfrak{C}_{n-1}^\eta$ to the set of n-clusters $\mathfrak{C}_n^\eta$. Realizing that any n-cluster $D \in \mathfrak{D}_n^\eta$ is a collection $(\mathcal{C}_i)$ of L_n-connected families of $(n-1)$-clusters, $\mathcal{C}_i = (C_i^s)$, we first introduce the domain of any such family $\mathcal{C}_i$ as $\mathrm{Dom}(\mathcal{C}_i) = \cup_s \mathrm{Dom}(C_i^s)$. Next, we define

$$\mathrm{Dom}(D) := \bigcup_i \mathrm{Dom}(\mathcal{C}_i) \cup \bigcup_{\alpha:\, \mathcal{K}_{n,\alpha}^\eta \nsim D} \mathrm{Dom}(\mathcal{K}_{n,\alpha}^\eta) \tag{61}$$

Furthermore, the length $|D|$ of the cluster is defined as

$$|D| := \sum_i |\mathcal{C}_i| = \sum_i \sum_s |C_i^s| \tag{62}$$

Note that this is possibly much smaller than the diameter of the cluster, since the sizes of the n-aggregates in the domain of D are not counted in the length of the cluster. The reason for this definition is that the cluster weights are not expected to be exponentially damped with the cluster diameter. Note, however, that the *probability* of a cluster to occur is exponentially damped with the size of the n-aggregates in its domain.

In the next proposition, we provide uniform bounds on the n-cluster weights. For the proof, see Section 11.

Proposition 8.3. There is $\beta_5 > 0$ such that for any $\beta \geqslant l_0\beta_5$, $\eta \in \Omega^{**}$, $\Lambda = \Lambda(N)$, $N \geqslant N^{**}(\eta)$, the inequalities

$$\sup_{x^*} \sum_{\substack{D \in \mathfrak{D}_n^\eta \\ x^* \in D}} \exp\left(\frac{\beta}{l_0}|D|\right) |\psi_n^\eta(D)| \leqslant 2^{-n} \quad n = 1, 2, \ldots \tag{63}$$

and

$$\sup_n \sup_{x^*} \sum_{\substack{C \in \mathfrak{C}_n^\eta \\ x^* \in C}} \exp\left(\frac{\beta}{l_0}|C|\right) |\phi_n^\eta(C)| \leqslant 1 \tag{64}$$

hold true.

Moreover, if $C \in \mathfrak{C}_n^\eta$ and $\eta'|_{\mathrm{Dom}(C)} = \eta|_{\mathrm{Dom}(C)}$, then also $C \in \mathfrak{C}_n^{\eta'}$ and $\phi_n^{\eta'}(C) = \phi_n^\eta(C)$. Similarly, $D \in \mathfrak{D}_n^\eta$ and $\eta'|_{\mathrm{Dom}(D)} = \eta|_{\mathrm{Dom}(D)}$ implies both $D \in \mathfrak{D}_n^{\eta'}$ and $\psi_n^{\eta'}(D) = \psi_n^\eta(D)$.

8.3. Expansion of Corner Aggregates

For any finite square $\Lambda = \Lambda(N)$ and $\eta \in \Omega$, all aggregates from the set $\cup_n \mathcal{K}_n^\eta$ are expanded in a finite number of steps. Afterwards, all corner aggregates are treated by a similar procedure. Throughout this section, we use the notation n_0 for the highest order in the collection of all normal aggregates. The expansion goes similarly as in the case of normal aggregates, so we only sketch it.

The renormalized weight of any compatible family of contours $\partial \subset \mathcal{K}_\infty^\eta$ is defined by the formula

$$\hat{\rho}^\eta(\partial) = \exp\left(-\sum_{\substack{C \in \mathfrak{C}_{n_0}^\eta \\ C \nsim \partial;\, |C| < 2l_\infty}} \phi_{n_0-1}^\eta(C)\right) \prod_{\Gamma \in \partial} \rho^\eta(\Gamma) \tag{65}$$

which factorizes over the corners, $\hat{\rho}^\eta(\partial) = \prod_i \hat{\rho}^\eta(\partial \cap \mathcal{K}_{\infty,i}^\eta)$, assuming $\Lambda(N)$ to be large enough. Clusters $C_1, C_2 \subset \mathfrak{C}_{n_0}^\eta$ are called ∞-incompatible whenever there is a corner aggregate $\mathcal{K}^{\eta}_{\infty,i}$ such that $C_1 \nsim \mathcal{K}_{\infty,i}$ and $C_2 \nsim \mathcal{K}_{\infty,i}$. Defining the weight $w^{\eta}(C)$ as

$$w^{\eta}(\mathcal{C}) = \frac{1}{\prod_i \hat{\mathcal{Z}}^{\eta}_{\infty,i}} \sum_{\partial \in \mathcal{D}^{\eta}_{\infty}} \hat{\rho}^{\eta}(\partial) \prod_{C \in \mathcal{C}^{\eta}_{n_0}} \left(e^{-\tilde{\phi}^{\eta}_{n_0}(C,\partial)} - 1 \right) \tag{66}$$

where

$$\hat{\mathcal{Z}}^{\eta}_{\infty,i} = \sum_{\partial \in \mathcal{D}^{\eta}_{\infty,i}} \hat{\rho}^{\eta}(\partial) \tag{67}$$

and

$$\tilde{\phi}^{\eta}_{\infty}(C, \partial) = \phi^{\eta}_{\infty}(C) \mathbf{1}_{\{C \nsim \partial;\ |C| \geqslant 2 l_{\infty}\}} \tag{68}$$

an obvious variant of Lemma 8.2 holds true and $w^{\eta}(\mathcal{C})$ factorizes into a product over maximal connected components of $\mathcal{C}$ w.r.t. ∞-incompatibility. Treating these as polymers in a new polymer model with ∞-incompatibility used as the incompatibility relation, and using the notation $\mathfrak{D}^{\eta}_{\infty}$ for the set of all clusters in this polymer model and $\psi^{\eta}_{\infty}(D)$ for the cluster weights, we obtain as the final step of the sequential expansion,

$$\mathcal{Z}^{\eta}_{n_0+1} = \exp\left(\sum_{D \in \mathfrak{D}^{\eta}_{\infty}} \psi^{\eta}_{\infty}(D) \right) \prod_i \hat{\mathcal{Z}}^{\eta}_{\infty,i} \tag{69}$$

Proposition 8.4. There exist constants $\beta_6 \geqslant \beta_5, c_6 > 0$ such that for any $\beta \geqslant l_0 \beta_6$, $\eta \in \Omega^{**}$, and volume $\Lambda(N)$, $N \geqslant N^{**}(\eta)$, one has the bound

$$\sum_{D \in \mathfrak{D}^{\eta}_{\infty}} |\psi^{\eta}_{\infty}(D)| \leqslant e^{-c_6 l_{\infty}} \tag{70}$$

Gathering all expansion steps, we arrive at the final expression for the partition function $\mathcal{Z}^{\eta}_{\Lambda}$ in the form

$$\begin{aligned} \log \mathcal{Z}^{\eta}_{\Lambda} = {} & -E^{\eta}(\emptyset) + \sum_{C \in \mathfrak{C}^{\eta}_0} \phi^{\eta}_0(C) + \sum_{n \geqslant 1} \sum_{D \in \mathfrak{D}^{\eta}_n} \psi^{\eta}_n(D) + \sum_{D \in \mathfrak{D}^{\eta}_{\infty}} \psi^{\eta}_{\infty}(D) \\ & + \sum_i \log \hat{\mathcal{Z}}^{\eta}_{\infty,i} + \sum_{n \geqslant 1} \sum_{\alpha} \log \hat{\mathcal{Z}}^{\eta}_{n,\alpha} \end{aligned} \tag{71}$$

The terms collected on the first line contain the "vacuum" energy under the boundary condition η, together with the contributions of clusters of all orders. Recall that the latter allow for a uniform exponential upper bound. On the second line there are the partition functions of all n- and all corner aggregates. Although we can provide only rough upper bounds for these terms, a crucial property to be used is that the probability of an aggregate to occur is exponentially small in the size of its boundary, see Section 7. In this sense, the above expansion is a natural generalization of the familiar "uniform" cluster expansion.(29)

8.4. Estimates on the Aggregate Partition Functions

In expression (71) we do not attempt to perform any detailed expansion of the aggregate's (log-)partition functions $\hat{\mathcal{Z}}^{\eta}_{n,\alpha}$ and $\hat{\mathcal{Z}}^{\eta}_{\infty,i}$ via a series of local and exponentially damped terms. Instead, we follow the idea that a locally ill-behaving boundary condition forces a partial coarse-graining represented above via the framework of aggregates of different orders. Although the detailed (cluster expansion-type) control within the aggregates is lost, we still can provide generic upper bounds on these partition functions. Notice a basic difference between n-aggregates and corner aggregates: The former contain only simple boundary contours the weights of which exponentially decay with the height of the contours. In some sense, the partition functions $\hat{\mathcal{Z}}^{\eta}_{n,\alpha}$ can be compared with the partition function of a 1d interface to get an upper bound. On the other hand, the corner aggregates are ensembles of contours the weight of which obey no uniform exponential bound with the space extension of the contours, and allow possibly for a non-trivial "degeneracy of vacuum". As a consequence, only rough (counting-type) estimates can be provided for the partition functions $\hat{\mathcal{Z}}^{\eta}_{\infty,i}$.

Lemma 8.5. There are constants $c_7, c_7' > 0$ ($c_7 \downarrow 0$ if $\beta \to \infty$) such that for any n-aggregate $\mathcal{K}^{\eta}_{n,\alpha}$, one has the bound

$$\log \hat{\mathcal{Z}}^{\eta}_{n,\alpha} \leqslant c_7 |\partial \mathcal{K}^{\eta}_{n,\alpha}| \tag{72}$$

For any corner aggregate $\mathcal{K}^{\eta}_{\infty,i}$,

$$\log \hat{\mathcal{Z}}^{\eta}_{\infty,i} \leqslant c_7' l_\infty^2 \tag{73}$$

9. ASYMPTOTIC TRIVIALITY OF THE CONSTRAINED GIBBS MEASURE ν_Λ^η

As the first application of expansion (71) we prove that the weak limit of the constrained measure ν_Λ^η coincides with the '+' phase Gibbs measure μ^+, finishing the first part of our program.

Proposition 9.1. There exists a constant $c>0$ such that for any $\beta \geqslant l_0\beta_6$ (with the β_6 the same as in Proposition 8.4), any $\eta \in \Omega^{**}$, and $X \subset \mathbb{Z}^2$ finite,

$$\|\nu_{\Lambda(N)}^\eta - \mu^+\|_X = O(e^{-cN}) \tag{74}$$

In particular, $\lim_{N\to\infty} \nu_{\Lambda(N)}^\eta = \mu^+$, $\boldsymbol{P}$-a.s.

Proof. The idea of the proof is to express the expectation $\nu_{\Lambda(N)}^\eta(f)$ of any local function f as the sum of a convergent series by using the multi-scale scheme developed in the last section, and to compare the series with a standard cluster expansion for $\nu_{\Lambda(N)}^{\eta\equiv+1}$. The difference between both series is given in terms of clusters both touching the boundary and the dependence set of f. Restricting only to the boundary conditions $\eta \in \Omega^{**}$ and volumes $\Lambda(N)$, $N \geqslant N^{**}(\eta)$ and using the exponential decay of the cluster weights, we prove the exponential convergence $\nu_{\Lambda(N)}^\eta(f) \to \nu^+(f)$.

For notational simplicity, we only restrict to a special case and give a proof of the equality

$$\lim_\Lambda \nu_\Lambda^\eta(\sigma_0=-1) = \mu^+(\sigma_0=-1) \tag{75}$$

The general case goes along the same lines.

Assuming $\sigma \in \Omega_\Lambda^+$, observe that $\sigma_0=-1$ if and only if the set $\mathcal{D}_\Lambda(\sigma)$ contains an odd number of contours Γ such that $0 \in \mathrm{Int}(\Gamma)$. In an analogy with (19), we write the ν_Λ^η-probability that $\sigma_0=-1$ in the form

$$\nu_\Lambda^\eta(\sigma_0=-1) = \frac{1}{\mathcal{Z}_\Lambda^\eta} \sum_{\Delta \sqsubset \Lambda} \mathcal{Z}_\Lambda^\eta(\backslash\Delta) \prod_{\Gamma\in\Delta} \rho^\eta(\Gamma) \tag{76}$$

where we have used the shorthand $\Delta \sqsubset \Lambda$ for any compatible family of contours in Λ such that card(Δ) is an odd integer and $0 \in \mathrm{Int}\,\Gamma$ for every $\Gamma \in \Delta$. Furthermore, $\mathcal{Z}_\Lambda^\eta(\backslash\Delta)$ is the partition function

$$\mathcal{Z}_\Lambda^\eta(\backslash\Delta) = \exp\left(-E_\Lambda^\eta(\emptyset)\right) \sum_{\partial\in\mathcal{D}_\Lambda(\backslash\Delta)} \prod_{\Gamma\in\partial} \rho^\eta(\Gamma) \tag{77}$$

of a polymer model over the restricted ensemble $\mathcal{K}_\Lambda(\backslash\Delta)\subset\mathcal{K}_\Lambda$ of all contours Γ such that (i) $\Gamma\sim\Delta$, and (ii) $0\notin\mathrm{Int}(\Gamma)$. We can now repeat the same procedure as in the last sections, but with the contour ensemble $\mathcal{K}_\Lambda$ being replaced by $\mathcal{K}^\eta(\backslash\Delta)$. A crucial observation is that all contours from the set $\mathcal{K}^\eta\setminus\mathcal{K}^\eta(\backslash\Delta)$ are balanced, at least for all $\eta\in\Omega^{**}$ and provided that the volume $\Lambda(N)$ is large enough. Hence, the sets of unbalanced contours coincide for both contour ensembles $\mathcal{K}^\eta$ and $\mathcal{K}^\eta(\backslash\Delta)$, hence, the same is true for the collections of both n- and corner aggregates. Finally, we compare the terms in the expansions for $\mathcal{Z}^\eta_\Lambda$ and $\mathcal{Z}^\eta_\Lambda(\backslash\Delta)$, and arrive at the formula

$$\log\frac{\mathcal{Z}^\eta_\Lambda(\backslash\Delta)}{\mathcal{Z}^\eta_\Lambda}=-\sum_{C\in\mathfrak{C}^\eta_0\backslash\mathfrak{C}^\eta_0(\backslash\Delta)}\phi^\eta_0(C)-\sum_{n\geqslant 1}\sum_{D\in\mathfrak{D}^\eta_n\backslash\mathfrak{D}^\eta_n(\backslash\Delta)}\psi^\eta_n(D)-\sum_{D\in\mathfrak{D}^\eta_\infty\backslash\mathfrak{D}^\eta_\infty(\backslash\Delta)}\psi^\eta_\infty(D) \tag{78}$$

where each of the three sums runs over all (0-, n-, or ∞-)clusters that are either incompatible with Δ or contain a contour Γ, $0\in\mathrm{Int}(\Gamma)$. By construction, each n-, respectively ∞-cluster is further required to be incompatible with an n-, respectively corner aggregate, and since their weights are uniformly exponentially bounded by Propositions 8.3 and 8.4, we get the uniform upper bound

$$\sup_\Lambda\left|\log\frac{\mathcal{Z}^\eta_\Lambda(\backslash\Delta)}{\mathcal{Z}^\eta_\Lambda}\right|\leqslant c|\Delta| \tag{79}$$

with a constant c large enough, as well as the existence of the limit

$$\lim_\Lambda\log\frac{\mathcal{Z}^\eta_\Lambda(\backslash\Delta)}{\mathcal{Z}^\eta_\Lambda}=-\sum_C{}'\,\phi_0(C) \tag{80}$$

where the sum runs over all finite 0-clusters in $\mathbb{Z}^2$ that are either incompatible with Δ or contain a contour surrounding the origin.

Since every $\Gamma\in\Delta$ surrounds the origin, it is necessarily balanced and satisfies $\rho^\eta(\Gamma)\leqslant\exp(-\frac{2\beta}{l_0}|\Gamma|)$. Combined with (79) and (80), one easily checks that

$$\lim_\Lambda\nu^\eta_\Lambda(\sigma_0=-1)=\sum_{\Delta\sqsubset\mathbb{Z}^2}\exp\left(-\sum_C{}'\,\phi_0(C)\right)\prod_{\Gamma\in\Delta}\rho(\Gamma) \tag{81}$$

and the convergence is exponentially fast. Obviously, the right-hand side coincides with the limit $\lim_\Lambda\mu^{\eta\equiv+1}_\Lambda(\sigma_0=-1)=\mu^+(\sigma_0=-1)$, which finishes the proof. ∎

10. RANDOM FREE ENERGY DIFFERENCE

In this section we analyze the limit behavior of the sequence of the random free energy differences

$$F_\Lambda^\eta = \log \mathcal{Z}_\Lambda^\eta - \log \mathcal{Z}_\Lambda^{-\eta} \tag{82}$$

In order to show that the probability that F_Λ^η takes a value in a fixed finite interval is bounded as $\mathcal{O}(N^{-\frac{1}{2}+\alpha})$ with $\alpha > 0$, we can use the local central limit upper bound proven in Appendix B, provided that a Gaussian-type upper bound on the characteristic functions of the random variables F_Λ^η can be established. The basic idea is to prove the latter by employing the sequential expansion for $\log \mathcal{Z}_\Lambda^\eta$ developed in Section 8 and by computing the characteristic functions in a neighborhood of the origin via a Mayer expansion. However, a technical problem arises here due to the high probability of the presence of corner aggregates. That is why we need to split our procedure in two steps that can be described as follows.

In the first step, we fix the boundary condition in the logarithmic neighborhood of the corners and consider the random free energy difference F_Λ^η conditioned on the fixed configurations. For this conditioned quantity a Gaussian upper bound on the characteristic function can be proven, implying a bound on the probability that the conditioned free energy difference $\boldsymbol{P}$-a.s. takes a value in a scaled interval (aN^δ, bN^δ). This can be combined with a Borel–Cantelli argument to exclude all values in such an interval, at least $\boldsymbol{P}$-a.s. and for all but finitely many volumes from a sparse enough sequence of volumes.

In the second step, we consider the contribution to the free energy difference coming from the corner aggregates. However, their contribution to the free energy will be argued to be of a smaller order when compared with the contribution of the non-corner terms.

Note that we also include the ∞-clusters in the first step. Because we have uniform bounds in η for the ∞-cluster weights, we are allowed to do so.

The free energy difference F_Λ^η can be computed by using the sequential expansion (71). For convenience, we rearrange the terms in the expansion by introducing

$$\begin{aligned} U^\eta(B) = &\sum_n \sum_\alpha \log \hat{\mathcal{Z}}_{n,\alpha}^\eta \mathbf{1}_{\{\mathrm{Dom}(\mathcal{K}_{n,\alpha}^\eta)=B\}} + \sum_i \log \hat{\mathcal{Z}}_{\infty,i}^\eta \mathbf{1}_{\{\mathrm{Dom}(\mathcal{K}_{\infty,i}^\eta)=B\}} \\ &+ \sum_C \phi_0^\eta(C) \mathbf{1}_{\{\mathrm{Dom}(C)=B\}} + \sum_n \sum_D \psi_n^\eta(D) \mathbf{1}_{\{\mathrm{Dom}(D)=B\}} \\ &+ \sum_D \psi_\infty^\eta(D) \mathbf{1}_{\{\mathrm{Dom}(D)=B\}} \end{aligned} \tag{83}$$

for any set $B \subset \partial\Lambda$. Note that any function $U^{\eta}(B)$ only depends on the restriction of η to the set $\underline{B}$. Using the notation $\bar{U}^{\eta}(B) = U^{\eta}(B) - U^{-\eta}(B)$, the expansion for the free energy difference F^{η}_{Λ} reads, formally,

$$F^{\eta}_{\Lambda} = 2\beta \sum_{x \in \underline{\partial\Lambda}} \eta_x + \sum_{B \subset \partial\Lambda} \bar{U}^{\eta}(B) \tag{84}$$

Obviously, no bulk contours contribute to $\bar{U}^{\eta}(B)$. Using the notation $\partial\Lambda_{C,i} := \{y^* \in \partial\Lambda : d[y^*, x^*_{C,i}] \leqslant 2l_{\infty}\}$ and $\partial\Lambda_C := \cup_{i=1}^{4} \partial\Lambda_{C,i}$, we consider the decomposition $F^{\eta}_{\Lambda} = \tilde{F}^{\eta}_{\Lambda} + \hat{F}^{\eta}_{\Lambda}$, where

$$\tilde{F}^{\eta}_{\Lambda} = 2\beta \sum_{x \in \underline{\partial\Lambda \setminus \partial\Lambda_C}} \eta_x + \sum_{\substack{B \subset \partial\Lambda \\ \mathrm{Dom}(B) \not\subset \partial\Lambda_C}} \bar{U}^{\eta}(B) \tag{85}$$

and

$$\hat{F}^{\eta}_{\Lambda} = 2\beta \sum_{x \in \underline{\partial\Lambda_C}} \eta_x + \sum_{\substack{B \subset \partial\Lambda \\ \mathrm{Dom}(B) \subset \partial\Lambda_C}} \bar{U}^{\eta}(B) \tag{86}$$

The first term, $\tilde{F}^{\eta}(B)$, can be analyzed by means of the Mayer expansion of its characteristic function

$$\begin{aligned} \tilde{\Psi}^{\eta}_{\Lambda}(t) := \boldsymbol{E}[\exp(it\tilde{F}^{\eta}_{\Lambda}) \mid \eta_{\underline{\partial\Lambda_C}}] &= \boldsymbol{E}\Bigg[\exp\Bigg(2it\beta \sum_{x \in \underline{\partial\Lambda \setminus \partial\Lambda_C}} \eta_x\Bigg) \\ &\quad \times \sum_{\mathcal{B}} \prod_{B \in \mathcal{B}} (e^{it\bar{U}^{\eta}(B)} - 1) \Bigg| \eta_{\underline{\partial\Lambda_C}}\Bigg] \\ &= [\Psi_0(t)]^{|\partial\Lambda \setminus \partial\Lambda_C|} \sum_{\mathcal{B}} w_t(\mathcal{B} \mid \eta_{\underline{\partial\Lambda_C}}) \end{aligned} \tag{87}$$

where we have assigned to any family $\mathcal{B}$ of subsets of the boundary the weight

$$\begin{aligned} w_t(\mathcal{B} \mid \eta_{\underline{\partial\Lambda_C}}) = \frac{1}{[\Psi_0(t)]^{|\partial\Lambda \setminus \partial\Lambda_C|}} \boldsymbol{E}\Bigg[&\exp\Bigg(2it\beta \sum_{x \in \underline{\partial\Lambda \setminus \partial\Lambda_C}} \eta_x\Bigg) \\ &\times \prod_{B \in \mathcal{B}} (e^{it\bar{U}^{\eta}(B)} - 1) \Bigg| \eta_{\underline{\partial\Lambda_C}}\Bigg] \\ &\times \mathbf{1}_{\{\forall B \in \mathcal{B} :\, B \not\subset \partial\Lambda_C\}} \end{aligned} \tag{88}$$

and have introduced the notation

$$\Psi_0(t) = \boldsymbol{E}[\exp(2it\beta\eta_0)] = \cos 2t\beta \tag{89}$$

Observing that

$$w(\mathcal{B}_1 \cup \mathcal{B}_2 \mid \underline{\eta_{\partial\Lambda_C}}) = w(\mathcal{B}_1 \mid \underline{\eta_{\partial\Lambda_C}})\, w(\mathcal{B}_2 \mid \underline{\eta_{\partial\Lambda_C}}) \tag{90}$$

whenever $B_1 \cap B_2 = \emptyset$ for any $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$, the last sum in Eq. (87) is a partition function of another polymer model and using the symbols $\mathfrak{B}, \mathfrak{B}_1, \ldots$ for the clusters in this model and w_t^T for the cluster weights, we get

$$\tilde{\Psi}_\Lambda^\eta(t) = [\Psi_0(t)]^{|\partial\Lambda \setminus \partial\Lambda_C|} \exp\left[\sum_{\mathcal{B}} w_t^T(\mathcal{B} \mid \underline{\eta_{\partial\Lambda_C}})\right] \tag{91}$$

A crucial observation is that for any $\eta \in \Omega^{**}$ and $\Lambda(N)$, $N \geqslant N^{**}(\eta)$ no corner aggregate contributes to the weight $w_t(\mathcal{B})$ for any $\mathcal{B}$. On the other hand, the partition function of any n-aggregate is balanced by a small probability of the aggregate to occur. Another observation is that every weight $w_t(\mathcal{B})$ is of order $\mathcal{O}(t^2)$ due to the symmetry of the distribution $\boldsymbol{P}$. To see this explicitly, formula (88) can be cast into a more symmetrized form,

$$\begin{aligned} w_t(\mathcal{B} \mid \underline{\eta_{\partial\Lambda_C}}) = \frac{1}{[\Psi_0(t)]^{|\operatorname{Supp}(\mathcal{B})|}} \boldsymbol{E}\left[T\left\{ t\left[2\beta \sum_{x \in \underline{\operatorname{Supp}(\mathcal{B})}} \eta_x + \frac{1}{2}\sum_{B\in\mathcal{B}} \bar{U}^\eta(B)\right]\right\}\right. \\ \left. \times \prod_{B\in\mathcal{B}} 2i\sin\left(\frac{t\bar{U}^\eta(B)}{2}\right) \mid \underline{\eta_{\partial\Lambda_C}}\right] \end{aligned} \tag{92}$$

where $\operatorname{Supp}(\mathcal{B}) := \cup_{B\in\mathcal{B}} B$ and

$$T\{Y\} := \begin{cases} i\sin Y & \text{if } \operatorname{card}(\mathcal{B}) = 2k-1 \\ \cos Y & \text{if } \operatorname{card}(\mathcal{B}) = 2k, \quad k \in \mathbb{N} \end{cases} \tag{93}$$

In Section 11.5 we give a proof of the following upper bound on the corresponding cluster weights:

Lemma 10.1. There exist constants $\beta_8, l_0 > 0$[3] such that for any $\beta \geqslant \beta_8 l_0$ there is $t_0 = t_0(\beta) > 0$ for which the following is true. For any $\eta \in \Omega^{**}$ and $\Lambda = \Lambda(N)$, $N \geqslant N^{**}(\eta)$, the inequality

[3]Recall that the construction of aggregates depends on the choice of l_0.

$$\sup_{x^*\in\partial\Lambda\backslash\partial\Lambda_C} \sum_{\mathcal{B}:\, x^*\in\mathrm{Supp}(\mathcal{B})} |w_t^T(\mathcal{B}\,|\,\eta_{\underline{\partial\Lambda_C}})| \leqslant \frac{1}{2}\beta^2 t^2 \tag{94}$$

is satisfied for all $|t|\leqslant t_0$.

With the help of the last lemma, it is easy to get an upper bound on $\tilde{\Psi}_\Lambda^\eta(t)$:

Lemma 10.2. Under the assumptions of Lemma 10.1, we have

$$\tilde{\Psi}_\Lambda^\eta(t) \leqslant \exp\left(-\frac{1}{2}\beta^2 t^2 |\partial\Lambda(N)\setminus\partial\Lambda_C(N)|\right) \tag{95}$$

for all $|t|\leqslant t_0$, $\eta\in\Omega^{**}$, and $N\geqslant N^{**}(\eta)$.

Proof. It immediately follows by combining Lemma 10.1, Eq. (91), and the bound $\Psi_0(t)\leqslant\exp[-\beta^2 t^2]$. ▮

For the corner part $\hat{F}_\Lambda^\eta$ of the free energy difference we use the next immediate upper bound:

Lemma 10.3. Given $\eta\in\Omega^{**}$ and $\beta\geqslant\beta_6 l_0$, then $\hat{F}_{\Lambda(N)}^\eta = \mathcal{O}(N^\delta)$ for any $\delta>0$.

Proof. Using Proposition 8.4 and Lemma 8.5, we have $\sum_{B\subset\partial\Lambda_C} |\bar{U}^\eta(B)| = \mathcal{O}(l_\infty^2)$ and the above claim immediately follows. ▮

Proof of Proposition 3.5. Combining Lemma 10.2 with Proposition B.1 in Appendix B, we get

$$\overline{\lim_{N\to\infty}}\, N^{\frac{1}{2}-\alpha} \boldsymbol{P}\left\{|\tilde{F}_{\Lambda(N)}^\eta| \leqslant N^\alpha\tau \,\middle|\, \eta_{\underline{\partial\Lambda_C}}\right\} < \infty \tag{96}$$

for any $\alpha,\tau>0$. By Lemma 10.3, $\tilde{F}$ can be replaced with the full free energy difference F. As a consequence,

$$\overline{\lim_{N\to\infty}}\, N^{\frac{1}{2}-\alpha} \boldsymbol{P}\{|F_{\Lambda(N)}^\eta|\leqslant\tau\} < \infty \tag{97}$$

and the proof is finished by applying Proposition 9.1. ▮

11. PROOFS

In this section, we collect the proofs omitted throughout the main text.

11.1. Proof of Proposition 7.5

In order to get the claimed exponential upper bound on the probability for an n-aggregate to occur, we need to analyze the way how the aggregates are constructed in more detail. We start with an extension of Definition 7.2. Throughout the section, a finite volume $\Lambda = \Lambda(N)$ is supposed to be fixed.

Definition 11.1. For every $n = 1, 2, \ldots,$ any maximal L_n-connected subset $\Delta \subset \mathcal{K} \setminus (\mathcal{K}_0^\eta \cup \mathcal{K}_1^\eta \cup \ldots \cup \mathcal{K}_{n-1}^\eta)$ is called an n-pre-aggregate.

Obviously, n-aggregates are exactly those n-pre-aggregates Δ that satisfy the condition $|\partial\Delta|_{\mathrm{con}} \leqslant l_n$. Moreover, every n-pre-aggregate can equivalently be constructed inductively by gluing pre-aggregates of lower orders:

Lemma 11.2. Every n-pre-aggregate Δ_n is the union of a family of $(n-1)$-pre-aggregates, $\Delta_n = \cup_\alpha \Delta_{n-1}^\alpha$. Moreover,

(i) Each $(n-1)$-pre-aggregate Δ_{n-1}^α satisfies $|\partial\Delta_{n-1}^\alpha|_{\mathrm{con}} > l_{n-1}$,

(ii) The family $(\Delta_{n-1}^\alpha)_\alpha$ is L_n-connected.

Proof. For $n = 1$ the statement is trivial.

Assume that $n \geqslant 2$, and let Δ_n be an n-pre-aggregate and $\Gamma \in \Delta_n$ be a contour. Then, there exists an $(n-1)$-pre-aggregate Δ_{n-1}^α such that $\Gamma \in \Delta_{n-1}^\alpha$ (otherwise Γ would be an element of a k-aggregate, $k \leqslant n-2$). Moreover, since Δ_{n-1}^α is not an $(n-1)$-aggregate by assumption, it satisfies $|\partial\Delta_{n-1}^\alpha|_{\mathrm{con}} > l_{n-1}$, proving (i).

The claim (ii) is obvious. ∎

Lemma 11.3. Let Δ by any family of unbalanced contours. Then,

(i) There exists a subset $\tilde{\Delta} \subset \Delta$ such that

(a) $\partial\tilde{\Delta} = \partial\Delta$,

(b) if $\Gamma_1, \Gamma_2, \Gamma_3 \in \tilde{\Delta}$ are any three mutually different contours, then $\partial\Gamma_1 \cap \partial\Gamma_2 \cap \partial\Gamma_3 = \emptyset$.

(ii) The inequality

$$\sum_{x \in \underline{\partial\Delta}} \eta_x < -\left(1 - \frac{4}{l_0}\right) |\partial\Delta| \tag{98}$$

holds true.

Proof. (i) Assume that $\Gamma_1, \Gamma_2, \Gamma_3 \subset \Delta$ is a triple of mutually different contours such that $\partial\Gamma_1 \cap \partial\Gamma_2 \cap \partial\Gamma_3 \neq \emptyset$. Since $\partial\Gamma_i$, $i = 1, 2, 3$ are connected subsets of $\partial\Lambda$, it is easy to realize that, up to a possible permutation of the index set $\{1,2,3\}$, one has $\partial\Gamma_1 \subset \partial\Gamma_2 \cup \partial\Gamma_3$. Hence, $\partial(\Delta \setminus \{\Gamma_1\}) = \partial\Delta$. Since the set Δ is finite, a subset $\tilde{\Delta} \subset \Delta$ with the claimed property is constructed by iterating the argument.

(ii) Let $\tilde{\Delta} \subset \Delta$ be the same as in (i). Then, using Lemma 5.2, the inclusion-exclusion principle implies

$$\begin{aligned}\sum_{x\in\underline{\partial\Delta}} \eta_x &= \sum_{\Gamma\in\tilde{\Delta}} \sum_{x\in\underline{\partial\Gamma}} \eta_x - \sum_{(\Gamma,\Gamma')\subset\tilde{\Delta}} \sum_{x\in\underline{\partial\Gamma\cap\partial\Gamma'}} \eta_x \\ &< -\left(1-\frac{2}{l_0}\right)\sum_{\Gamma\in\tilde{\Delta}}|\partial\Gamma| + \sum_{(\Gamma,\Gamma')\subset\tilde{\Delta}}|\partial\Gamma\cap\partial\Gamma'| \\ &\leqslant -\left(1-\frac{4}{l_0}\right)|\partial\Delta| \quad \blacksquare\end{aligned} \tag{99}$$

It remains to prove that one still gets a large deviation upper bound by replacing the sum over the boundary sites $x\in\underline{\partial\Delta}$ in Eq. (98) with the sum over all $x\in\underline{\mathrm{Con}(\partial\Delta)}$, provided that Δ is a pre-aggregate. Technically, we need to exploit the basic feature of any pre-aggregate Δ that the set $\mathrm{Con}(\partial\Delta)\setminus\partial\Delta$ is not "too big". A minor complication lies in the fact that the boundary distance $d[\partial\gamma,\partial\gamma']$ is allowed to exceed the contour distance $d[\gamma,\gamma']$. To overcome this difficulty, it is useful to define

$$\widetilde{\mathrm{Con}}(\partial\Delta) = \partial\Delta \cup \left\{x\in\mathrm{Con}(\partial\Delta);\ \forall\gamma\in\Delta : d[x,\partial\gamma] > \frac{|\partial\gamma|}{l_0}\right\} \tag{100}$$

for which the first equation in the proof of Lemma 5.2 implies the upper bound

$$|\partial\Delta|_{\mathrm{con}} \leqslant \left(1+\frac{2}{l_0}\right)|\widetilde{\mathrm{Con}}(\partial\Delta)| \tag{101}$$

We are now ready to prove the following key estimate from which Proposition 7.5 immediately follows by using a large deviation upper bound.

Lemma 11.4. Let Δ be an n-pre-aggregate, $n=1,2,\ldots$ Then,

$$\sum_{x\in\underline{\mathrm{Con}(\partial\Delta)}} \eta_x \leqslant -\frac{1}{3}|\partial\Delta|_{\mathrm{con}} \tag{102}$$

uniformly in n.

Proof. We prove by induction in the order of the pre-aggregates the refined bound

$$\sum_{x\in\underline{\widetilde{\mathrm{Con}}(\partial\Delta)}} \eta_x \leqslant -\left(1-3\sum_{i=1}^{n}\frac{L_i}{l_{i-1}}\right)|\widetilde{\mathrm{Con}}(\partial\Delta)| \tag{103}$$

for any n-pre-aggregate Δ, from which the statement follows by using the definition 7.1 of length scales l_n and L_n, and Eq. (101). Indeed, one obtains then

$$\begin{aligned}\sum_{x\in\mathrm{Con}(\partial\Delta)} \eta_x &\leqslant -\left(1-3\sum_{i=1}^{\infty}\frac{L_i}{l_{i-1}}\right)|\widetilde{\mathrm{Con}}(\partial\Delta)| + |\,\mathrm{Con}(\partial\Delta)\setminus\widetilde{\mathrm{Con}}(\partial\Delta)| \\ &\leqslant -\left(\frac{1}{4}-\frac{2}{l_0}\right)\frac{|\partial\Delta|_{\mathrm{con}}}{1+\frac{2}{l_0}} \leqslant -\frac{1}{3}|\partial\Delta|_{\mathrm{con}}\end{aligned} \tag{104}$$

First, assume that Δ is a 1-pre-aggregate, and let $\Delta=\cup_{i=1}^{m}A_i$ be the (unique) decomposition of Δ into disjoint subsets such that $\partial\Delta=\cup_{i=1}^{m}\partial A_i$ is the decomposition of $\partial\Delta$ into maximal connected components. For convenience, we use the notation $J_i:=\partial A_i$. Considering furthermore the decomposition $\widetilde{\mathrm{Con}}(\partial\Delta)\setminus\partial\Delta=\cup_{k=1}^{m-1}G_k$ into maximal connected components, the set $\widetilde{\mathrm{Con}}(\partial\Delta)$ can be finally written as the union

$$\widetilde{\mathrm{Con}}(\partial\Delta)=\left(\bigcup_{i=1}^{m}J_i\right)\bigcup\left(\bigcup_{k=1}^{m-1}G_k\right) \tag{105}$$

of disjoint connected subsets, which satisfy the inequalities $|J_i|>l_0$ and $|G_k|\leqslant L_1$, for all $i,k=1,2,\ldots$ Using Lemma 11.3, we have $\sum_{x\in\underline{J_i}}\eta_x\leqslant-\left(1-\frac{4}{l_0}\right)|J_i|$, and since $\sum_{k=1}^{m-1}|G_k|\leqslant\frac{L_1}{l_0}\sum_{i=1}^{m}|J_i|$, we finally get

$$\begin{aligned}\sum_{x\in\underline{\widetilde{\mathrm{Con}}(\partial\Delta)}} \eta_x &= \sum_{i=1}^{m}\sum_{x\in\underline{J_i}}\eta_x+\sum_{k=1}^{m-1}|G_k|\leqslant-\left(1-\frac{L_1+4}{l_0}\right)\frac{|\widetilde{\mathrm{Con}}(\partial\Delta)|}{1+\frac{L_1}{l_0}} \\ &\leqslant -\left(1-\frac{3L_1}{l_0}\right)|\widetilde{\mathrm{Con}}(\partial\Delta)|\end{aligned} \tag{106}$$

provided that, say, $L_1\geqslant 4$.

Next, we will prove the statement for an arbitrary n-pre-aggregate Δ. By Lemma 11.2, Δ is the union of a family of $(n-1)$-pre-aggregates, $\Delta=\cup_i\Delta_{n-1}^i$. In order to generalize our strategy used in the $n=1$ case, we consider the (possibly disconnected) boundary sets $J_i=\widetilde{\mathrm{Con}}(\partial\Delta_{n-1}^i)$, and the family of connected sets $(G_i)_{i=1,2,\ldots}$ defined as the maximal connected components of the set $\mathrm{Con}(\partial\Delta) \setminus \cup_i \mathrm{Con}(\Delta^i_{n-1})$. Note that $\#\{G_i\} = \#\{J_i\} - 1$ and the identity $\widetilde{\mathrm{Con}}(\Delta) = (\cup_i J_i) \cup (\cup_i G_i)$. Hence, by using the induction hypothesis,

$$\sum_{x \in \underline{\widetilde{\mathrm{Con}}(\partial\Delta)}} \eta_x = \sum_{i=1}^{m} \sum_{x \in \underline{J_i}} \eta_x + \sum_{k=1}^{m-1} |G_k| \leqslant \left[-\left(1 - 3\sum_{i=1}^{n-1} \frac{L_i}{l_{i-1}}\right) + \frac{L_n}{l_{n-1}} \right] \frac{|\widetilde{\mathrm{Con}}(\partial\Delta)|}{1 + \frac{L_n}{l_{n-1}}}$$
$$\leqslant -\left(1 - 3\sum_{i=1}^{n} \frac{L_i}{l_{i-1}}\right) |\widetilde{\mathrm{Con}}(\partial\Delta)|, \tag{107}$$

as required. ∎

11.2. Proof of Proposition 8.3

The proof goes by induction in the order of aggregates.

The case $n = 1$. As the initial step we bound the sums over 1-clusters in $\mathfrak{D}^\eta_1$. Recall that the 1-clusters consist of 0-clusters which connect 1-aggregates $\mathcal{K}^\eta_{1,\alpha}$. Throughout this section we use the shorthand $\tilde{\beta} := \frac{\beta}{l_0}$.

From Proposition 5.4 we know that for any integer r_0,

$$\sum_{\substack{C \in \mathfrak{C}^\eta_0:\ |C| \geqslant r_0 \\ C \ni x}} |\phi^\eta_0(C)| \exp(\tilde{\beta}(2 - (1/8))|C|) \leqslant 1$$

which implies

$$\sum_{\substack{C \in \mathfrak{C}^\eta_0:\ |C| \geqslant r_0 \\ C \ni x}} |\phi^\eta_0(C)| \exp(2\tilde{\beta}(1 - (1/8))|C|) \leqslant \exp(-\tilde{\beta} r_0/8) \tag{108}$$

We split the procedure into four steps as follows.

Part 1. For any 1-cluster in $\mathfrak{D}^\eta_1$, none of its 0-clusters contributes to the dressed weight of a 1-aggregate. Hence, all these 0-clusters have at least size L_1. Moreover, they are incompatible with a 1-aggregate $\mathcal{K}^\eta_{1,\alpha}$. Using Lemma 5.2 and choosing $r_0 = L_1$ in (108), this results in the inequality

$$\sum_{\substack{C \in \mathfrak{C}^\eta_0 \\ C \not\sim \mathcal{K}^\eta_{1,\alpha}}} |\phi^\eta_0(C)| \exp(2\tilde{\beta}(1 - (1/8))|C|) \leqslant l_1^2 \exp\left[-(\tilde{\beta} L_1)/8\right] \leqslant 2^{-2} \tag{109}$$

Part 2. In order to prove the convergence of the cluster expansion resulting from the Mayer expansion, we apply Proposition A.2. As our initial estimate, we get, using (109) and since

$$C \overset{1}{\not\leftrightarrow} C' \;\Leftrightarrow\; \exists\alpha \quad \text{such that } C, C' \not\sim \mathcal{K}^{\eta}_{1,\alpha}$$

the inequality

$$\begin{aligned}
&\sum_{\substack{C\in\mathfrak{C}^{\eta}_0\\ C\overset{1}{\leftrightarrow}C_0}} |\phi^{\eta}_0(C)| \exp(2\tilde{\beta}(1-(1/8))|C|)\\
&\quad\leqslant \sum_{\mathcal{K}^{\eta}_{1,\alpha}:\ \mathcal{K}^{\eta}_{1,\alpha}\not\sim C_0}\ \sum_{\substack{C\in\mathfrak{C}^{\eta}_0\\ C\not\sim\mathcal{K}^{\eta}_{1,\alpha}}} \exp(2\tilde{\beta}(1-(1/8))|C|)|\phi^{\eta}_0(C)| \qquad (110)\\
&\quad\leqslant 2^{-2}\#\{\mathcal{K}^{\eta}_{1,\alpha}\not\sim C_0\}
\end{aligned}$$

Part 3. Using Lemma 8.2, the weight of any set of 0-clusters appearing in the Mayer expansion is bounded as

$$|w^{\eta}_1(\mathcal{C})| \leqslant \prod_{C\in\mathcal{C}} (e^{|\phi^{\eta}_0(C)|}-1) \leqslant \prod_{C\in\mathcal{C}} 2|\phi^{\eta}_0(C)|$$

Hence, by using Proposition A.2, we obtain the bound

$$\sum_{\substack{C_1\overset{1}{\not\leftrightarrow}C_0\\ C_1\in\mathfrak{D}^{\eta}_1}} |\psi^{\eta}_1(C_1)| \exp[(2\tilde{\beta}(1-(1/8))-1/2)|C_1|] \leqslant 2^{-1}\#\{\mathcal{K}^{\eta}_{1,\alpha}\not\sim C_0\} \qquad (111)$$

Taking now $C_0\in\mathfrak{C}^{\eta}_0$ such that $\mathcal{K}^{\eta}_{1,\alpha}$ is the only 1-aggregate satisfying $C_0 \not\sim \mathcal{K}^{\alpha}_1$, inequality (111) yields

$$\sum_{\substack{C_1\not\sim\mathcal{K}^{\eta}_{1,\alpha}\\ C_1\in\mathfrak{D}^{\eta}_1}} |\psi^{\eta}_1(C_1)| \exp[(2\tilde{\beta}(1-(1/8))-1/2)|C_1|] \leqslant 2^{-1} \qquad (112)$$

Part 4. In order to bound the sum over all 1-clusters $C_1\in\mathfrak{D}^{\eta}_1$ such that $C_1\ni x$ and $|C_1|\geqslant r_1$, we use that $|C_1|\geqslant L_1$ and write

$$\begin{aligned}
&\sum_{\substack{C_1\ni x,\ |C_1|\geqslant r_1\\ C_1\in\mathfrak{D}^{\eta}_1}} |\psi^{\eta}_1(C_1)| \exp[(2\tilde{\beta}(1-(1+1/2)/8)-1/2)|C_1|]\\
&\quad\leqslant \sum_{\mathcal{K}^{\eta}_{1,\alpha}}\ \sum_{\substack{C_1\not\sim\mathcal{K}^{\eta}_{1,\alpha}:\ C_1\ni x\\ |C_1|\geqslant r_1,\ C_1\in\mathfrak{D}^{\eta}_1}} |\psi^{\eta}_1(C_1)| \exp[(2\tilde{\beta}(1-(1+1/2)/8)-1/2)|C_1|]
\end{aligned} \qquad (113)$$

Substituting (112), we obtain

$$(113) \leqslant \sum_{\mathcal{K}^{\eta}_{1,\alpha}} 2^{-1} \exp(-(\tilde{\beta}/8) \cdot \max[d(\mathcal{K}^{\eta}_{1,\alpha}, x), r_1])$$

$$\leqslant \sum_{R=0}^{\infty} \sum_{\mathcal{K}^{\eta}_{1,\alpha}:\ d(\mathcal{K}^{\eta}_{1,\alpha},x)=R} 2^{-1} \exp(-\epsilon\tilde{\beta} \cdot \max[R, r_1]) \tag{114}$$

The last sum can be estimated by a partial integration and we finally get

$$(113) \leqslant \exp(-\tilde{\beta} r_1/8) \left[r_1^2 + \frac{16 r_1}{\tilde{\beta}} + 2 \right] \leqslant 2^{-1} \cdot 4 r_1^2 \exp(-\tilde{\beta} r_1/8)$$

where we have used that $r_1 \geqslant L_1$ and that L_1 is large enough.

Induction step. The induction hypothesis reads

$$\sum_{\substack{C_i \ni x:\ |C_i| \geqslant r_i \\ C_i \in \mathfrak{D}^{\eta}_i}} |\psi^{\eta}_i(C_i)| \exp\left[\left(2\tilde{\beta} \left(1 - \sum_{j=0}^{i+1} (1/2)^j/8 \right) - \sum_{j=1}^{i} (1/2)^j \right) |C_i| \right]$$
$$\leqslant 4 \cdot 2^{-i} r_i^2 \exp(-\tilde{\beta}(1/2)^{i+1} r_i/8) \tag{115}$$

for any $1 \leqslant i \leqslant n-1$.

Part 1. As in part 1 of the $n=1$ case, we want to prove first that

$$\sum_{\substack{C \in \mathfrak{C}^{\eta}_{n-1} \\ C \not\sim \mathcal{K}^{\eta}_{n,\alpha}}} |\phi^{\eta}_{n-1}(C)| \exp\left[\left(2\tilde{\beta} \left(1 - \sum_{j=0}^{n} (1/2)^j/8 \right) - \sum_{j=1}^{n-1} (1/2)^j \right) |C| \right] \leqslant 2^{-n-1} \tag{116}$$

Recalling definition (58) for $\phi^{\eta}_{n-1}(C)$, we know that $\phi^{\eta}_{n-1}(C) = \psi^{\eta}_j(C)$ for any $C \in \mathfrak{D}^{\eta}_j$. Hence, using (115) with $r_i = L_n$, we write

$$(116) \leqslant l_n^2 \sum_{i=1}^{n-1} \sum_{\substack{C_i \ni x:\ |C_i| \geqslant L_n \\ C_i \in \mathfrak{D}^{\eta}_i}} |\psi^{\eta}_i(C_i)| \exp\left[\left(2\tilde{\beta} \left(1 - \sum_{j=0}^{n} (1/2)^j/8 \right) - \sum_{j=1}^{n-1} (1/2)^j \right) |C_i| \right]$$
$$\leqslant 4 l_n^2 L_n^2 \sum_{i=0}^{n-1} 2^{-i} \exp\left[\left(-\tilde{\beta} \sum_{j=i+1}^{n} (1/2)^j/4 - \sum_{j=i+1}^{n-1} (1/2)^j \right) L_n \right]$$
$$\leqslant 2^{-n-1} \cdot 32 l_n^2 L_n^2 \exp\left[-\tilde{\beta}(1/2)^n L_n/4 \right] \leqslant 2^{-n-1} \tag{117}$$

where we have used that $l_n = \exp(L_n/2^n)$ and $\tilde{\beta}$ is large enough. This proves inequality (116).

Part 2. Similarly as in the $n=1$ case, we prove by using (116) the inequality

$$\sum_{\substack{C \overset{n}{\nleftrightarrow} C_0 \\ C \in \mathfrak{C}_{n-1}^{\eta}}} |\phi_{n-1}^{\eta}(C)| \exp\left[\left(2\tilde{\beta}\left(1 - \sum_{j=0}^{n}(1/2)^j/8\right) - \sum_{j=1}^{n-1}(1/2)^j\right)|C|\right]$$
$$\leqslant 2^{-n-1}\#\{\mathcal{K}_{n,\alpha}^{\eta} \not\sim C_0\}$$

Part 3. By construction, any n-cluster $C_n \in \mathfrak{D}_n^{\eta}$ consists of a family of 0-clusters $C_0 \in \mathfrak{C}_0^{\eta}$ and i-clusters $C_1 \in \mathfrak{D}_0^{\eta}$, $0 \leqslant i \leqslant n-1$, which are all incompatible with $\mathcal{K}_n^{\eta}$. Using Lemma 8.2 again, we have the upper bound

$$|w_n^{\eta}(C_n)| \leqslant \prod_{i=0}^{n-1} \prod_{C \in C_n \cap \mathfrak{D}_i^{\eta}} 2|\psi_i^{\eta}(C)|$$

where we have identified $\psi_0^{\eta}(.) \equiv \phi_0^{\eta}(.)$ and $\mathfrak{D}_0^{\eta} \equiv \mathfrak{C}_0^{\eta}$. Applying Proposition A.2 with $z(C) = 2|\psi_i^{\eta}(C)|$ then gives

$$\sum_{\substack{C_n \overset{n}{\nleftrightarrow} C_0 \\ C_n \in \mathfrak{D}_n^{\eta}}} |\psi_n^{\eta}(C_n)| \exp\left[\left(2\tilde{\beta}\left(1 - \sum_{j=0}^{n}(1/2)^j/8\right) - \sum_{j=1}^{n}(1/2)^j\right)|C|\right]$$
$$\leqslant 2^{-n}\#\{\mathcal{K}_{n,\alpha}^{\eta} \not\sim C_0\}$$

Taking again $C_0 \in \mathfrak{C}_0^{\eta}$ such that $\mathcal{K}_{n,\alpha}^{\eta} \not\sim C_0$ implies the inequality

$$\sum_{\substack{C_n \not\sim \mathcal{K}_{n,\alpha}^{\eta} \\ C_n \in \mathfrak{D}_n^{\eta}}} |\psi_n^{\eta}(C_n)| \exp\left[\left(2\tilde{\beta}\left(1 - \sum_{j=0}^{n}(1/2)^j/8\right) - \sum_{j=1}^{n}(1/2)^j\right)|C_n|\right] \leqslant 2^{-n} \tag{118}$$

Part 4. Repeating the argument for the $n=1$ case, we obtain the inequality

$$\sum_{\substack{C_n \ni x,\ |C_n| \geqslant r_n \\ C_n \in \mathfrak{D}_n^{\eta}}} |\psi_n^{\eta}(C_n)| \exp\left[\left(2\tilde{\beta}\left(1 - \sum_{j=0}^{n+1}(1/2)^j/8\right) - \sum_{j=1}^{n}(1/2)^j\right)|C_n|\right]$$
$$\leqslant 2^{-n} \cdot 4r_n^2 \exp(-(1/2)^{n+1}\tilde{\beta}r_n/8) \tag{119}$$

Using that $r_n \geqslant L_n$ for any $C_n \in \mathfrak{D}_n^\eta$ and choosing $r_n = L_n$ proves the proposition for the weights ψ_n^η, $n = 1, 2, \dots$

Equation (58) reads that $\phi_n^\eta(C) = \psi_j^\eta(C)$ whenever $C \in \mathfrak{D}_j^\eta$ and $j \leqslant n$. Using further that $\mathfrak{C}_n^\eta = \mathfrak{C}_0^\eta \cup \mathfrak{D}_1^\eta \cup \dots \cup \mathfrak{D}_n^\eta$ and summing up the cluster weights of the clusters of all orders yields inequality (64), which finishes the proof.

11.3. Proof of Proposition 8.4

Let n_0 be the same as in Section 8.3. Due to the second part of Proposition 8.3,

$$\sup_x \sum_{\substack{C \ni x,\ |C| \geqslant r_0 \\ C \in \mathfrak{C}_{n_0}^\eta}} |\phi_{n_0}^\eta(C)| \exp[(\beta/4l_0)]\, |C| \leqslant 2 \exp(-(3\beta/4l_0) r_0)$$

According to the definition of the corner-aggregates, we have

$$\sum_{\substack{C \not\sim \mathcal{K}_{\infty,i} \\ C \in \mathfrak{C}_{n_0}^\eta}} |\phi_{n_0}^\eta(C)| \exp[(\beta/4l_0)|C|] \leqslant 2 l_\infty^2 \exp(-l_\infty(3\beta/2l_0)) \leqslant 2^{-3} \exp(-l_\infty(\beta/l_0))$$

Applying Proposition A.2, we obtain

$$\sum_{\substack{C \not\sim \mathcal{K}_{\infty,i} \\ C \in \mathfrak{D}_\infty^\eta}} |\psi_\infty^\eta(C)| \exp[(\beta/4l_0)]\, |C| \leqslant 2^{-2} \exp(-l_\infty(\beta/l_0))$$

which implies

$$\sum_{D \in \mathfrak{D}_\infty^\eta} |\psi_\infty^\eta(D)| \leqslant \exp(-(3\beta/2l_0) l_\infty)$$

11.4. Proof of Lemma 8.5

Let $\eta \in \Omega^{**}$ and $\mathcal{K}_{n,\alpha}^\eta$ be an n-aggregate, $n = 1, 2, \dots$ Recall that

$$\hat{\mathcal{Z}}_{n,\alpha}^\eta = \sum_{\partial \in \mathcal{D}_{n,\alpha}^\eta} \hat{\rho}^\eta(\partial) \tag{120}$$

where

$$\hat{\rho}^\eta(\partial) = \prod_{\Gamma \in \partial} \rho^\eta(\Gamma) \exp\left(- \sum_{\substack{C \in \mathfrak{C}_{n-1}^\eta \\ C \not\sim \partial;\ |C| < L_n}} \phi_{n-1}^\eta(C) \right) \tag{121}$$

Using the η-uniform bounds

$$\rho^{\eta}(\Gamma)\leqslant \exp[-2\beta(|\Gamma|-|\partial\Gamma|)] \tag{122}$$

and

$$\sup_{x^\star}\sum_{\substack{C\in\mathfrak{C}^{\eta}_{n-1}\\ x^\star\in C}}|\phi^{\eta}_{n-1}(C)|\leqslant \exp\left[-\frac{3\beta}{l_0}\right] \tag{123}$$

for all $n=1,2,\ldots$, one can subsequently write (for simplicity, we use the shorthand $\varepsilon=\exp\left[-\frac{3\beta}{l_0}\right]$ below):

$$\begin{aligned}
\hat{\mathcal{Z}}^{\eta}_{n,\alpha}&\leqslant \sum_{\partial\in\mathcal{D}^{\eta}_{n,\alpha}}\prod_{\Gamma\in\partial}\exp[-(2\beta-\varepsilon)|\Gamma|+2\beta|\partial\Gamma|]\\
&\leqslant e^{(\varepsilon+4e^{-2\beta+\varepsilon})|\partial\mathcal{K}^{\eta}_{n,\alpha}|}\sum_{\partial\in\mathcal{D}^{\eta}_{n,\alpha}}\prod_{\Gamma\in\partial}\exp[-(2\beta-\varepsilon)|\Gamma|+(2\beta-\varepsilon-4e^{-2\beta+\varepsilon}))|\partial\Gamma|]\\
&\leqslant e^{(\varepsilon+4e^{-2\beta+\varepsilon})|\partial\mathcal{K}^{\eta}_{n,\alpha}|}\sum_{\partial\in\mathcal{D}^{\eta}_{n,\alpha}}\prod_{\Gamma\in\partial}\prod_{\gamma\in\Gamma}\exp[-(2\beta-\varepsilon)|\gamma|+(2\beta-\varepsilon-4e^{-2\beta+\varepsilon})|\partial\gamma|]\\
&\leqslant e^{(\varepsilon+4e^{-2\beta+\varepsilon})|\partial\mathcal{K}^{\eta}_{n,\alpha}|}\left\{1+\sum_{\gamma\ni p}\exp[-(2\beta-\varepsilon)|\gamma|+(2\beta-\varepsilon-4e^{-2\beta+\varepsilon}))|\partial\gamma|]\right\}^{|\partial\mathcal{K}^{\eta}_{n,\alpha}|}
\end{aligned} \tag{124}$$

where the last sum runs over all pre-contours (= connected components of contours) such that a fixed dual bond $p=\langle x,y\rangle^\star$, $d(x,\Lambda^c)=d(y,\Lambda^c)=1$ is an element of γ and it is the leftmost bond with these properties, w.r.t. a fixed orientation on the boundary. To estimate this sum, we associate with each pre-contour γ a path (= sequence of bonds; not necessarily unique) starting at p. Every such a path consists of steps choosing from three of in total four possible directions. One easily realizes that, for every such a path, the total number of steps to the right is bounded from below by $|\partial\gamma|$. Hence, the last sum in (124) is upper-bounded via the summation over all paths started at p, so that to each step going to the right (respectively to the left/up/down) one assigns the weight $e^{-4e^{-2\beta+\varepsilon}}$ (respectively $e^{-2\beta+\varepsilon}$), which yields

$$\begin{aligned}
&\sum_{\gamma\ni p}\exp[-(2\beta-\varepsilon)|\gamma|+(2\beta-\varepsilon-4e^{-2\beta+\varepsilon}))|\partial\gamma|]\\
&\qquad\leqslant e^{-2\beta+\varepsilon}\sum_{n=1}^{\infty}\left(2e^{-2\beta+\varepsilon}+e^{-4e^{-2\beta+\varepsilon}}\right)^n\leqslant 2e^{-2\beta+\varepsilon}
\end{aligned} \tag{125}$$

All in all, one obtains

$$\hat{\mathcal{Z}}^{\eta}_{n,\alpha} \leqslant e^{(\varepsilon+6e^{-2\beta+\varepsilon})|\partial\mathcal{K}^{\eta}_{n,\alpha}|} \tag{126}$$

proving the first part of the statement.

The proof of the second part is trivial by counting the number of all configurations in the square volume with side $2l_\infty$. Note that the latter contains all contours $\Gamma\in\partial$ for any configuration $\partial\in\mathcal{D}^{\eta}_{\infty,i}$ and that the weights of all clusters renormalizing the contour weights are summable due to Proposition 8.3.

11.5. Proof of Lemma 10.1

Due to Proposition A.2, it is enough to show that the inequality

$$\sum_{\mathcal{B}:\, x^*\in\mathrm{Supp}(\mathcal{B})} |w_t(\mathcal{B}\,|\,\eta_{\underline{\partial\Lambda_C}})| \exp\left(\frac{1}{2}\beta^2t^2|\,\mathrm{Supp}(\mathcal{B})|\right) \leqslant \frac{1}{2}\beta^2t^2 \tag{127}$$

holds true for all $|t|\leqslant t_0$, with a constant $t_0>0$. Remark that the RHS of the last equation is not optimal and can be improved, as obvious from the computation below.

In order to prove (127), we use the symmetric representation (92) of the weight $w_t(\mathcal{B}\,|\,\eta_{\underline{\partial\Lambda_C}})$, the lower bound $\Psi_0(t)\geqslant e^{-\alpha}$ which is true for any $\alpha>0$ provided that $|t|\leqslant t_1(\alpha)$ with a constant $t_1(\alpha)>0$, and the estimate

$$\begin{aligned}
&\left|T\left\{t\left[2\beta\sum_{x\in\underline{\mathrm{Supp}(\mathcal{B})}}\eta_x+\frac{1}{2}\sum_{B\in\mathcal{B}}\bar{U}^{\eta}(B)\right]\right\}\right| \\
&\quad\leqslant\left\{\begin{cases} t\left[2\beta\sum_{x\in B}|\eta_x|+\frac{1}{2}|\bar{U}^{\eta}(B)|\right] & \text{for } \mathcal{B}=\{B\} \\ 1 & \text{otherwise}\end{cases}\right.
\end{aligned} \tag{128}$$

which will be enough in order to get the t^2 factor in what follows. Using Proposition 8.3 and Lemma 8.5, we get a uniform upper bound $|\bar{U}^{\eta}(B)|\leqslant c|B|$ with a constant $c>0$ such that $c\downarrow 0$ for $\beta\uparrow\infty$. Hence, in the case $\mathcal{B}=\{B\}$ we have

$$\begin{aligned}
&|w_t(\mathcal{B}=\{B\})\,|\,\eta_{\underline{\partial\Lambda_C}}| \\
&\quad\leqslant e^{\alpha|B|}t^2\boldsymbol{E}\left[\left(2\beta\sum_{x\in B}|\eta_x|+\frac{1}{2}\sum_{B\in\mathcal{B}}|\bar{U}^{\eta}(B)|\right)\prod_{B\in\mathcal{B}}|\bar{U}^{\eta}(B)|\,\Big|\,\eta_{\underline{\partial\Lambda_C}}\right] \\
&\quad\leqslant e^{\alpha|B|}t^2\left(2\beta+\frac{c}{2}\right)|B|\,\boldsymbol{E}\left[|\bar{U}^{\eta}(B)|\,\big|\,\eta_{\underline{\partial\Lambda_C}}\right]
\end{aligned} \tag{129}$$

Note that the above uniform upper bound on $|\bar{U}^\eta(B)|$ is not sufficient to get a sensible estimate on the conditional expectation. However, a more detailed upper bound can be obtained. Without loss of generality, we can assume that $B\cap\partial\Lambda_C=\emptyset$, so that the conditioning on $\eta_{\underline{\partial\Lambda_C}}$ can be omitted. First, assume there is an aggregate[4] $\mathcal{K}_\alpha^\eta$ such that $\mathrm{Dom}(\mathcal{K}_\alpha^\eta)=B$. Then, Lemma 8.5 gives the estimate $\log\hat{\mathcal{Z}}_\alpha^\eta\leqslant c_7|B|$ and, since $|\partial\mathcal{K}_\alpha^\eta|\geqslant|B|/2$, Proposition 7.5 reads that the probability of such an event is bounded by $\exp(-\frac{c_5}{2}|B|)$. Second, assume there is a family of aggregates $(\mathcal{K}_{\alpha_i}^\eta)_i$ (of possibly different orders) such that $D^\eta:=\cup_i\mathrm{Dom}(\mathcal{K}_{\alpha_i}^\eta)\subset B$. Then, any cluster C such that $\mathrm{Dom}(C)=B$ has the length $|C|\geqslant|B\setminus D^\eta|$ and Proposition 8.3 gives the estimate

$$\sum_{\substack{C\in\cup_n\mathfrak{C}_n^\eta\\ \mathrm{Dom}(C)=B}}|\phi_n^\eta|\leqslant\exp\left(-\frac{\beta}{2l_0}|B\setminus D^\eta|\right)$$

Moreover, the probability that $D^\eta=D$ for a fixed set D is bounded by $e^{-\frac{c_5}{2}|D|}$. Note, however, that the above two scenarios are possible only provided that $|B|\geqslant l_1$, otherwise we only get a contribution from 0-clusters, the sum of which is bounded by $e^{-\frac{\beta}{2l_0}|B|}$. All in all, we obtain

$$\begin{aligned}\boldsymbol{E}\left[|U^\eta(B)|\right] &\leqslant c_7|B|\,e^{-\frac{c_5}{2}|B|}\mathbf{1}_{|B|\geqslant l_1}+e^{-\frac{\beta}{2l_0}|B|}+\mathbf{1}_{|B|\geqslant l_1}\sum_{D\subset B}e^{-\frac{c_5}{2}|D|-\frac{\beta}{2l_0}|B\setminus D|}\\ &\leqslant e^{-\frac{\beta}{2l_0}|B|}+\mathbf{1}_{|B|\geqslant l_1}(c_7+1)|B|\,e^{-\frac{c_5}{4}|B|}\end{aligned}\tag{130}$$

provided that β/l_0 is large enough. Recall that c_5 does not depend on l_0, which means that the latter can be adjusted as large as necessary. Using the same argument for $U^{-\eta}(B)$ and substituting (130) into (129), we get

$$\begin{aligned}\sum_{\substack{B\ni x\\ B\subset\partial\Lambda}}|w_t(\mathcal{B}=\{B\})\,|\,\eta_{\underline{\partial\Lambda_C}})|\,e^{\tau|B|}&\leqslant 2\cdot t^2(2\beta+\frac{c}{2})\sum_{\substack{B\ni x\\ B\subset\partial\Lambda}}|B|\,e^{(\tau+\alpha)|B|}\\ &\quad\times\left[e^{-\frac{\beta}{2l_0}|B|}+\mathbf{1}_{|B|\geqslant l_1}(c_7+1)|B|\,e^{-\frac{c_5}{4}|B|}\right]\leqslant\tau'\beta t^2\end{aligned}\tag{131}$$

which is true for any $\tau'>0$ provided that τ and α are chosen sufficiently small and l_0 (and hence l_1) sufficiently large. This argument can easily be generalized by taking into account all collections $\mathcal{B}$, $\mathrm{card}(\mathcal{B})>1$. Hence, the proof of (127) is completed by choosing $\tau=\frac{1}{2}\beta^2t^2$, under the condition $|t|\leqslant t_0$ with $t_0=t_0(\beta)$ being small enough.

[4] For simplicity, we suppress the subscript n here.

12. CONCLUDING REMARKS AND SOME OPEN QUESTIONS

Our result that a typical boundary condition (w.r.t. a symmetric distribution) suppresses both mixed and interface states explains why these states are typically not observed in experimental situations without a special preparation. To a certain extent it justifies the standard interpretation of extremal invariant Gibbs measures as pure phases.

Although this result, which finally solves the question raised in ref. 39, is only about the two-dimensional Ising ferromagnet, and thus seemingly of limited interest, it is our opinion that the perturbation approach developed in the paper is actually very robust (compare ref. 24). As we have observed at various points in the paper, there seems to be no barrier except some technical ones to extend the analysis to the Ising model with random boundary conditions in higher dimensions. In fact, there might be extensions of our approach into various different directions. In particular, both the random distribution of the boundary terms and the phase transition itself could lack the plus–minus symmetry, and one might also consider a more general Pirogov–Sinai set-up in which the number of extremal Gibbs measures could be larger than two. Another possible extension could be to finite-range Hopfield-type models, in which periodic or fixed boundary conditions lack a coherence property with respect to the possible Gibbs measures, and thus are expected to behave as random ones.[44] Actually, our result can be translated in terms of the Mattis (= single-pattern Hopfield) model with fixed boundary conditions, proving the chaotic size-dependence there.

More generally, in principle the phenomenon of the exclusion of interface states for typical boundary conditions might well be of relevance for spin glass models of Edwards–Anderson type, which has indeed been one of our main motivations. Our result illustrates in a simple way how the Newman–Stein metastate program, designed for the models exhibiting the chaotic size-dependence, can be realized. The number of states, as well as the number of "physically relevant" states for short-range spin glasses has been an issue of contention for a long time. In this paper, we have provided a very precise distinction between the set of all Gibbs states, the set of all extremal Gibbs measures, and the set of "typically visible" ones, without restricting *a priori* to the states with a particular symmetry. We hope the provided criterion might prove useful in a more general context.

We mention that the restriction to sparse enough sequences of volumes is essential to obtain almost sure results. Actually, for a regular sequence of volumes, we expect all mixtures (in dimension three all translation-invariant Gibbs measures) to be almost sure limit points, although in a null-recurrent way. This still would mean that the metastate would not be affected, and that it would be concentrated on the plus and minus measures. See also the discussion in ref. 16. However, proving this conjecture goes beyond the presented technique and remains an open question.

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APPENDIX A. CLUSTER MODELS

In this section we present a variant of the familiar result on the convergence of the cluster expansion for polymer models, which proves useful in the cases when the summation over polymers becomes difficult because of their high geometrical complexity. Such a situation arises, for example, in the applications of the cluster expansion to the study of the convergence of high-temperature (Mayer) series in lattice models with an infinite-range potential. Since the Mayer expansion techniques are by no means restricted to the high-temperature regimes (note e.g. its application in the RG schemes for low-temperature contour models), the result below can be applied in a wide class of problems under a perturbation framework. In our context, we use the result to provide upper bounds on the weights ψ_n^η of n-clusters, see Section 11.2.

We consider an abstract cluster model defined as follows. Let $G = (S, \nsim)$ be a finite or countable non-oriented graph and call its vertices *polymers*. Any two polymers $X \nsim Y$ are called *incompatible*, otherwise they are *compatible*, $X \sim Y$. By convention, we add the relations $X \nsim X$ for all $X \in S$. Any non-empty finite set $\Delta \subset S$ is called a *cluster* whenever there exists no decomposition $\Delta = \Delta_1 \cup \Delta_2$ such that Δ_1 and Δ_2 are non-empty disjoint sets of polymers and $\Delta_1 \sim \Delta_2$, where the latter means that $X \sim Y$ for all $X \in \Delta_1$ and $Y \in \Delta_2$. Let $\mathcal{P}(S)$ denote the set of all finite subsets of S and $\mathcal{C}(S)$ denote the set of all clusters. A function $g : \mathcal{P}(S) \mapsto \mathbb{C}$ is called a *weight* whenever

(i) $g(\emptyset) = 1$,

(ii) If $\Delta_1 \sim \Delta_2$, then $g(\Delta_1 \cup \Delta_2) = g(\Delta_1)\, g(\Delta_2)$.

If the extra condition

(iii) $g(\Delta) = 0$ whenever there is an $X \in \Delta$ such that $X \nsim \Delta \setminus \{X\}$

holds true, then we obtain the familiar *polymer model*. In the sequel we do not assume Condition (iii) to be necessarily true, unless stated otherwise.

Note a simple duality between the classes of polymer and cluster models: Any cluster model over the graph $G=(S,\nsim)$ is also a polymer model over the graph $G'=(\mathcal{C}(S),\nsim)$. The other inclusion is also trivially true. A natural application of this duality is to the polymer models with a complicated nature of polymers. Such polymers can often be represented as clusters in a new cluster model with the polymers being simpler geometric objects.

To any set $A\in\mathcal{P}(S)$ we assign the *partition function* $Z(A)$ by

$$Z(A)=\sum_{\Delta\subset A} g(\Delta) \tag{A.1}$$

The map between the functions g and Z is actually a bijection and the last equation can be inverted by means of the Möbius inversion formula. In particular, we consider the function $g^T:\mathcal{P}(S)\mapsto\mathbb{C}$ such that the Möbius conjugated equations

$$\log Z(A)=\sum_{\Delta\subset A} g^T(\Delta), \qquad g^T(\Delta)=\sum_{A\subset\Delta}(-1)^{|\Delta\setminus A|}\log Z(A) \tag{A.2}$$

hold true for all $A\in\mathcal{P}(S)$ and $\Delta\in\mathcal{P}(S)$, respectively. The function g^T is called a *cluster* weight, the name being justified by the following simple observation:

Lemma A.1. For any cluster model, $g^T(\Delta)=0$ whenever Δ is not a cluster.

A familiar result about the polymer model is the exponential decay of the cluster weight g^T under the assumption on a sufficient exponential decay of the weight g, see refs. 29 and 35. We use the above duality to extend this result to the cluster models, formulating a new condition that can often be easily checked in applications.

Proposition A.2. Let positive functions $a,b:S\mapsto\mathbb{R}^+$ be given such that either of the following conditions is satisfied:

(1) (*Polymer model*)
Condition (iii) is fulfilled and[5]

$$\sup_{X\in S}\frac{1}{a(X)}\sum_{Y\nsim X} e^{(a+b)(Y)}|g(Y)|\leqslant 1 \tag{A.3}$$

[5]We use the convention $\frac{0}{0}=0$ here.

(2) (*Cluster model*)
There is $z: S \mapsto \mathbb{R}^+$ satisfying the condition

$$\sup_{X \in S} \frac{1}{a(X)} \sum_{Y \nsim X} e^{(2a+b)(Y)} z(Y) \leqslant 1 \tag{A.4}$$

such that $|g(\Delta)| \leqslant \prod_{X \in \Delta} z(X)$ for all $\Delta \in \mathcal{P}(S)$.

Then,

$$\sup_{X \in S} \frac{1}{a(X)} \sum_{\Delta \nsim X} e^{\sum_{Y \in \Delta} b(Y)} |g^T(\Delta)| \leqslant 1 \tag{A.5}$$

Proof. (1) For the case of the polymer models, see ref. 29 or better ref. 35 for the proof.

(2) To prove the statement for a cluster model, we represent it as a polymer model over the graph $(\mathcal{C}(S), \nsim)$ and make use of the above result. Hence, it is enough to show the inequality

$$\sum_{\substack{\Delta \in \mathcal{C}(S) \\ \Delta \nsim X}} e^{\sum_{Y \in \Delta} (a+b)(Y)} |g(\Delta)| \leqslant a(X) \tag{A.6}$$

for all $X \in S$. Indeed, then one gets

$$\sum_{\substack{\Delta \in \mathcal{C}(S) \\ \Delta \nsim \Delta_0}} e^{\sum_{Y \in \Delta} (a+b)(Y)} |g(\Delta)| \leqslant \sum_{Y \in \Delta_0} a(Y) \tag{A.7}$$

for all $\Delta_0 \in \mathcal{C}(S)$ and the statement about the polymer models yields

$$\sum_{\substack{\Delta^* \in \mathcal{C}(\mathcal{C}(S)) \\ \Delta^* \nsim \Delta_0}} e^{\sum_{\Delta \in \Delta^*} \sum_{Y \in \Delta} b(Y)} |g^T(\Delta^*)| \leqslant \sum_{Y \in \Delta_0} a(Y) \tag{A.8}$$

where the sum on the LHS is over all clusters incompatible with Δ_0 in the polymer model with the set of polymers $\mathcal{C}(S)$. Since the weights $g^T(\Delta)$ of the clusters in the original cluster model are related to the cluster weights $g^T(\Delta^*)$ in the polymer model under consideration as

$$g^T(\Delta) = \sum_{\Delta^*:\ \cup_{\Delta' \in \Delta^*} \Delta' = \Delta} g^T(\Delta^*) \tag{A.9}$$

we immediately get

$$\sum_{\Delta \not\sim X} e^{\sum_{Y\in\Delta} b(Y)} |g^T(\Delta)| \leqslant \sum_{\substack{\Delta^* \in \mathcal{C}(\mathcal{C}(S)) \\ \Delta^* \not\sim X}} e^{\sum_{\Delta\in\Delta^*} \sum_{Y\in\Delta} b(Y)} |g^T(\Delta^*)| \leqslant a(X) \quad \text{(A.10)}$$

which is inequality (A.5).

Using the notation $\hat{z}(X) := z(X)\, e^{a(X)+b(X)}$ and

$$\mathfrak{z}_X(A) = \sum_{\substack{\Delta\in\mathcal{C}(A) \\ \Delta\ni X}} \prod_{Y\in\Delta} \hat{z}(Y) \quad \text{(A.11)}$$

for any $A \in \mathcal{P}(S)$ and $X \in A$, inequality (A.6) follows from the next two lemmas. ∎

Lemma A.3. The function $\mathfrak{z}_X(A)$ satisfies the recurrence inequality

$$\mathfrak{z}_X(A) \leqslant \hat{z}(X) \exp\left[\sum_{\substack{Y\not\sim X \\ Y\in A\setminus\{X\}}} \mathfrak{z}_Y(A\setminus\{X\}) \right] \quad \text{(A.12)}$$

Proof. For any cluster Δ we split $\Delta \setminus \{X\}$ into connected components, i.e. a family of clusters (Δ_j), and subsequently write:

$$\begin{aligned}
\mathfrak{z}_X(A) &= \hat{z}(X) \sum_{\substack{\Delta\in\mathcal{C}(A) \\ \Delta\subset A\setminus\{X\}}} \prod_j \prod_{Y\in\Delta_j} \hat{z}(Y) \\
&\leqslant \hat{z}(X) \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\substack{Y_1,\dots,Y_n\in A\setminus\{X\} \\ \forall j:\ Y_j\not\sim X}} \prod_{j=1}^{n} \sum_{\substack{\Delta_j\subset A\setminus\{X\} \\ \Delta_j\ni Y_j}} \prod_{Y\in\Delta_j} \hat{z}(Y) \qquad \text{(A.13)} \\
&= \hat{z}(X) \sum_{n=0}^{\infty} \frac{1}{n!} \left[\sum_{\substack{Y\not\sim X \\ Y\in\mathcal{A}\setminus\{X\}}} \mathfrak{z}_Y(A\setminus\{X\}) \right]^n \\
&= \hat{z}(X) \exp\left[\sum_{\substack{Y\not\sim X \\ Y\in A\setminus\{X\}}} \mathfrak{z}_Y(A\setminus\{X\}) \right] \quad ∎
\end{aligned}$$

Lemma A.4. Assume that

$$\sum_{Y\not\sim X} \hat{z}(Y) e^{a(Y)} \leqslant a(X) \quad \text{(A.14)}$$

Then

$$\sum_{Y \not\sim X} \mathfrak{z}_Y(S) \leqslant a(X) \tag{A.15}$$

Proof. We prove the inequality

$$\mathfrak{z}_X(A) \leqslant \hat{z}(X) e^{a(X)} \tag{A.16}$$

for all $A \in \mathcal{P}(S)$ and $X \in A$, by induction in the number of polymers in the set A. Assuming that this bound is satisfied whenever $|A| < n$, we can estimate $\mathfrak{z}_X(A)$ for $|A| = n$ by using Lemma A.3, condition (A.14), and the induction hypothesis as follows:

$$\mathfrak{z}_X(A) \leqslant \hat{z}(X) \exp\left[\sum_{Y \not\sim X} \hat{z}(Y) e^{a(Y)}\right] \leqslant \hat{z}(X) e^{a(X)} \tag{A.17}$$

As the statement is obvious for $|A| = 1$, the lemma is proven. ∎

APPENDIX B. INTERPOLATING LOCAL LIMIT THEOREM

We present here a simple general result that can be useful in the situations where a full local limit theorem statement is not available due to the lack of detailed control on the dependence among random variables the sum of which is under consideration. For a detailed explanation of the central and the local limit theorems as well as the analysis of characteristic functions in the independent case, see e.g. ref. 13. Here, under only mild assumptions, we prove an asymptotic upper bound on the probabilities in a regime that interpolates between the ones of the central and the local limit theorem. Namely, we have the following result that is a simple generalization of Lemma 5.3 in ref. 16:

Proposition B.1. Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of random variables and denote by $\psi_n(t)$ the corresponding characteristic functions, $\psi_n(t) = \mathbb{E}\, e^{itX_n}$. If $(A_n)_{n \in \mathbb{N}}$, $(\delta_n)_{n \in \mathbb{N}}$ and $(\tau_n)_{n \in \mathbb{N}}$ are strictly positive sequences of reals satisfying the assumptions

(i) $\overline{\lim}_{n \to \infty} A_n \int_{-\tau_n}^{\tau_n} dt\, |\psi_n(t)| \leqslant 2\pi$

(ii) There is $k > 1$ such that $\lim_{n \to \infty} \frac{A_n}{\delta_n^k \tau_n^{k-1}} = 0$

then

$$\overline{\lim_{n\to\infty}} \frac{A_n}{\delta_n} \boldsymbol{P}\{a\delta_n \leqslant X_n \leqslant b\delta_n\} \leqslant b-a \tag{B.1}$$

for any $a<b$.

Remark B.2. Note that:

(1) Up to a normalization factor, Condition (i) of the proposition only requires A_n to be chosen as

$$A_n = \mathcal{O}\left(\left[\int_{-\tau_n}^{\tau_n} dt\, |\psi_n(t)|\right]^{-1}\right) \tag{B.2}$$

(2) If there is ε_1 such that $A_n \tau_n \leqslant n^{\varepsilon_1}$ eventually in n, then Assumption (ii) of the proposition is satisfied whenever $\delta_n \tau_n \geqslant n^{\varepsilon_2}$ with a constant $\varepsilon_2 > 0$.

(3) The choice $\delta_n = A_n$ (if available) gives an upper-bound on the probabilities in the regime of the central limit theorem. On the other hand, $\delta_n = \text{const}$ corresponds to the regime of the local limit theorem. However, for the latter choice it can be difficult to check the assumptions, and that is why one has to allow for a sufficient scaling of δ_n, see Part (2) of this remark.

(4) Much more information about the distribution of the random variables X_n would be needed in order to get any *lower bounds* on the probabilities (except for the case $\tau_n = \infty$ in which a full local limit theorem can be proven). This is a hard problem that we do not address here.

Proof. Let sequences $(A_n), (\tau_n), (\delta_n)$ be given such that the assumption of the proposition is true and take an arbitrary positive function $h \in C^\infty(\mathbb{R})$ for which (i) $h(x)=0$ for any $x \notin (-\varepsilon, \varepsilon)$ and (ii) $\int_{-1}^{1} dx\, h(x) = 1$. Using the notation G_n for the distribution function of X_n, we consider its "regularized version" $\bar{G}_n$ defined by the Lebesgue density

$$\frac{d\bar{G}_n(x)}{dx} = \int_{-\infty}^{\infty} dG_n(y)\, h_n(x-y) \tag{B.3}$$

where $h_n(x) := \frac{1}{\delta_n} h(\frac{x}{\delta_n})$. Obviously, $\frac{d\bar{G}_n}{dx} \in C^\infty(\mathbb{R})$ and it can be expressed by the Fourier integral as follows:

$$\begin{aligned}\frac{d\bar{G}_n(x)}{dx} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, e^{-itx}\psi_n(t)\int_{-\infty}^{\infty} dy\, e^{ity} h_n(y)\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, e^{-itx}\psi_n(t)\,\hat{h}(t\delta_n)\end{aligned}\tag{B.4}$$

where $\hat{h}(t) := \int_{-\infty}^{\infty} dx\, e^{itx} h(x)$ and we have used that $\psi_n(t)\,\hat{h}(t\delta_n) \in L^1(\mathbb{R})$ following from Assumption (i) of the proposition and from the bounds $|\psi_n(t)|, |\hat{h}(t)| \leqslant 1$. Moreover, if $k > 1$ is such that Assumption (ii) holds, then, using the bound $|\hat{h}(t)| \leqslant c|t|^{-k}$ which is true with some constant c for all $t \in \mathbb{R}\setminus\{0\}$, we obtain the estimate

$$\begin{aligned}\overline{\lim_{n\to\infty}}\sup_x A_n\frac{dG_n(x)}{dx} &\leqslant \frac{1}{2\pi}\,\overline{\lim}_{n\to\infty}\\ &\left(A_n\int_{-\tau_n}^{\tau_n} dt\,|\psi_n(t)| + \int_{\mathbb{R}\setminus[-\tau_n,\tau_n]} \mathrm{d}t\,|\hat{h}(t\delta_n)|\right)\\ &\leqslant 1 + \frac{1}{\pi}\frac{c}{k-1}\lim_{n\to\infty}\frac{A_n}{\delta_n^k\tau_n^{k-1}} = 1\end{aligned}\tag{B.5}$$

Finally, by using the inequality

$$\begin{aligned}\boldsymbol{P}\{a\delta_n \leqslant X_n \leqslant b\delta_n\} &= \int_{a\delta_n}^{b\delta_n} dG_n(y)\int_{-\infty}^{\infty} dx\, h_n(x-y)\\ &\leqslant \int_{(a-\varepsilon)\delta_n}^{(b+\varepsilon)\delta_n} dx \int_{a\delta_n}^{b\delta_n} dG_n(y)\, h_n(x-y)\\ &\leqslant \int_{(a-\varepsilon)\delta_n}^{(b+\varepsilon)\delta_n} d\bar{G}_n(x)\end{aligned}\tag{B.6}$$

we get

$$\lim_{n\to\infty}\frac{A_n}{\delta_n}\boldsymbol{P}\{a\delta_n \leqslant X_n \leqslant b\delta_n\} \leqslant (b-a+2\varepsilon)\quad \lim_{n\to\infty}\sup_x A_n\frac{dG_n(x)}{dx} \leqslant b-a+2\varepsilon \tag{B.7}$$

and the proposition follows by taking the limit $\varepsilon \to 0$. ∎

REFERENCES

1. M. Aizenman, Translation invariance and instability of phase coexistence in the two dimensional Ising model, *Commun. Math. Phys.* **73**:83–94 (1980).

2. L. Bertini, E. N. M. Cirillo, and E. Olivieri, Graded cluster expansion for lattice systems (2004), mp_arc 04-207.
3. L. Bertini, E. N. M. Cirillo, and E. Olivieri, Renormalization Group in the uniqueness region: Weak Gibbsianity and convergence (2004), mp_arc 04-208.
4. C. Borgs and J. Z. Imbrie, A unified approach to phase diagrams in field theory and statistical mechanics, *Commun. Math. Phys.* **123**:305–328 (1989).
5. C. Borgs and R. Kotecký, Surface-induced finite-size effects for the first-order phase transition, *J. Stat. Phys.* **79**:43–116 (1995).
6. A. Bovier, Statistical mechanics of disordered systems, *MaPhySto Lecture Notes* 10 (Aarhus, 2001).
7. A. Bovier, A. C. D. van Enter, and B. Niederhauser, Stochastic symmetry breaking in a Gaussian Hopfield model, *J. Stat. Phys.* **95**:181–213, 1999. See also B. Niederhauser's thesis "Mathematical Aspects of Hopfield Models" (2000) at www.math.tu-berlin.de/stoch/Kolleg/homepages/Niederhauser.
8. A. Bovier and V. Gayrard, Hopfield models as generalized random mean-field models, in *Mathematical Aspects of Spin Glasses and Neural Networks*, in A. Bovier and P. Picco, eds. (Birkhäuser, Boston, 1998), pp. 3–89.
9. J. Bricmont and A. Kupiainen, Phase transitions in the 3d random field Ising model, *Commun. Math. Phys.* **116**:539–572 (1988).
10. J. Bricmont and A. Kupiainen, High temperature expansions and dynamical systems, *Commun. Math. Phys.* **178**:703–732 (1996).
11. J. Bricmont and J. Slawny, Phase transitions in systems with a finite number of dominant ground states. *J. Stat. Phys.* **54**:89–161, 1989.
12. H. von Dreifus, A. Klein, and J. F. Perez, Taming Griffiths' singularities: Infinite differentiability of quenched Correlation functions, *Commun. Math. Phys.* **170**:21–39 (1995).
13. R. Durrett. *Probability: Theory and Examples* (Wadsworth, Inc., Belmont, 1985).
14. A. C. D. van Enter, stiffness exponent, number of pure states and Almeida–Thouless Line in spin-glasses, *J. Stat. Phys.* **60**:275–279 (1990).
15. A. C. D. van Enter, R. Fernández, A. Sokal, Regularity properties and pathologies of position-space renormalization-group transformations: Scope and limitations of Gibbsian theory, *J. Stat. Phys.* **72**:879–1167 (1993).
16. A. C. D. van Enter, I. Medved', and K. Netočný, Chaotic size dependence in the Ising model with random boundary conditions. *Markov Proc. Rel. Fields* **8**:479–508 (2002).
17. A. C. D. van Enter and H.G. Schaap, Infinitely many states and stochastic symmetry in a Gaussian Potts–Hopfield model, *J. Phys. A* **35**:2581–2592 (2002).
18. D. S. Fisher, J. Fröhlich, and T. Spencer, The Ising model in a random magnetic field, *J. Stat. Phys.* **34**:863–870 (1984).
19. D. S. Fisher and D. A. Huse, Pure states in spin glasses, *J. Phys. A, Math. Gen.* **20**:L997–L1003 (1987).
20. J. Fröhlich, Mathematical aspects of disordered systems, *Les Houches Session XLIII Course 9* (1984).
21. J. Fröhlich and J. Z. Imbrie, Improved perturbation expansion for disordered systems: Beating Griffiths singularities. *Commun. Math. Phys.* **96**:145–180 (1984).
22. G. Gallavotti, Phase separation line in the two-dimensional Ising model. *Commun. Math. Phys.* **27**:103–136 (1972).
23. H.O. Georgii, *Gibbs Measures and Phase Transitions*, (de Gruyter, Berlin, 1988).
24. For those readers who feel that we have invoked a heavy machinery to prove something which is physically quite plausible, we have some sympathy, but as a partial justification we'd like to quote from the recent book by G. Gallavotti, F. Bonetti and G. Gentile, Aspects of Ergodic, Qualitative and Statistical Theory of Motion, (Springer, 2004) in

particular what they have to say about cluster expansion methods (p. 257):

> The proliferation of alternative or independent and different proofs, or of nontrivial extensions, shows that in reality the problem is a natural one and that the methods of studying it with the techniques of this section are also natural although they are still considered by many as not elegant (and not really natural) and they are avoided when possible or it is said that "it must be possible to obtain the same result in a simpler way" (often not followed by any actual work in this direction).

25. R. Haag, The algebraic approach to quantum statistical mechanics, equilibrium states and hierarchy of stability, In *Critical Phenomena, Sitges School Proceedings, Springer Lecture Notes of Physics*, Vol. 54: (1976), pp. 155–188.
26. Y. Higuchi, On the absence of non-translationally invariant Gibbs states for the two-dimensional Ising system, in *Random Fields: Rigorous Results in Statistical Mechanics and Quantum Field Theory (Esztergom 1979)*, J. Fritz, J. L. Lebowitz, and D. Szász, eds (North-Holland, Amsterdam, 1981).
27. W. Holsztynski and J. Slawny, Peierls condition and the number of ground states, *Commun. Math. Phys.* **61**:177–190 (1978).
28. J. Imbrie, The ground state of the three-dimensional random-field Ising model. *Commun. Math. Phys.* **98**:146–176 (1985).
29. R. Kotecký and D. Preiss, Cluster expansions for abstract polymer models, *Commun. Math. Phys.* **103**:491–498 (1986).
30. C. Külske, Metastates in disordered mean-field models: random field and Hopfield models. *J. Stat. Phys.* **88**:1257–1293 (1997).
31. C. Külske, Limiting behavior of random Gibbs measures: Metastates in some disordered mean-field models. in *Mathematical Aspects of Spin Glasses and Neural Networks*, A. Bovier and P. Picco, eds (Birkhäuser, Boston, 1998), pp. 151–160.
32. C. Külske, Metastates in disordered mean-field models ii, *J. Stat. Phys.* **91**:155–176 (1998).
33. C. Külske, Gibbs measures of disordered spin systems, WIAS preprint no. 653 (2001).
34. M. Loebl, Ground state incongruence in 2D spin glasses Revisited, *Electron. J. Combin.* **11**, R40 (2004). (arXiv:math.CO/032101.)
35. S. Miracle-Solé, On the convergence of cluster expansions, *Physica A* **279**:244–249 (2000).
36. H. Narnhofer and D. W. Robinson, Stability of pure thermodynamic phases in quantum statistics, *Commun. Math. Phys.* **41**:89–97 (1975).
37. C. M. Newman, *Topics in Disordered Systems*, Lectures in Mathematics ETH-Zürich (Birkhäuser, Basel, 1997).
38. C. M. Newman and D. L. Stein, Multiple states and thermodynamic limits in short-ranged Ising spin-glass models, *Phys. Rev. B* **46**:973–982 (1992).
39. C. M. Newman and D. L. Stein, Metastate approach to thermodynamic chaos, *Phys. Rev. E* **55**:5194–5211 (1997).
40. C. M. Newman and D. L. Stein, Thermodynamic chaos and the structure of short-range spin glasses in *Mathematical Aspects of Spin Glasses and Neural Networks*, A. Bovier and P. Picco, eds (Birkhäuser, Boston, 1998), pp. 243–287.
41. C. M. Newman and D. L. Stein, The state(s) of replica symmetry breaking: Mean field theories vs. short-ranged spin glasses, *J. Stat. Phys.* **106**:213–244 (2002). Formerly known as: "Replica Symmetry Breaking's New Clothes".
42. C. M. Newman and D. L. Stein, Ordering and broken symmetry in short-ranged spin glasses, *J. Phys.: Condens. Matter* **15**(32):1319–1364 (2003).

43. C. M. Newman and D. L. Stein, Nonrealistic behavior of mean-field spin glasses, *Phys. Rev. Lett.* **91**:197–205 (2003).
44. H. Nishimori, D. Sherrington, and W. Whyte, Finite-dimensional neural networks storing structured patterns, *Phys. Rev. E* **51**:3628–3642 (1995).
45. S. A. Pirogov and Ya. G. Sinai, Phase diagrams of classical lattice systems. *Theor. Math. Phys.* **25**:1185–1192 (1976).
46. S. A. Pirogov and Ya. G. Sinai, Phase diagrams of classical lattice systems, Continuation *Theor. Math. Phys.* **26**:39–49 (1976).
47. M. Talagrand, *Spin Glasses: A Challenge for Mathematicians* (Springer-Verlag, 2003).
48. M. Zahradník, An alternate version of Pirogov–Sinai theory, *Commun. Math. Phys.* **93**:559–581 (1984).